MATH 241-02, Multivariable Calculus, Spring 2019

Section 9.4: The Cross Product

In the last section we learned one way to multiply vectors, namely the dot product, which produces a scalar (a number). The dot product is particularly useful for calculating the angle between two vectors and for determining when two vectors are perpendicular (dot product of zero). In this section we learn another way to multiply vectors, known as the **cross product**. Unlike the dot product, the result of the cross product is a vector.

The Cross Product

We begin with the geometric definition of the cross product, denoted $\mathbf{v} \times \mathbf{w}$. Then, we will explain some key properties of the cross product and use them to derive an algebraic formula for $\mathbf{v} \times \mathbf{w}$.

Cross Product: If $\mathbf{v}$ and $\mathbf{w}$ are three-dimensional vectors, then

$$\mathbf{v} \times \mathbf{w} \;=\; (|\mathbf{v}|\,|\mathbf{w}|\sin\theta)\,\mathbf{n}\,,$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$, and $\mathbf{n}$ is the unit normal vector orthogonal to the plane determined by $\mathbf{v}$ and $\mathbf{w}$ using the right-hand rule.

Note that the cross product of two vectors is another **vector** and that the length of this vector is $|\mathbf{v}|\,|\mathbf{w}|\sin\theta$ because the length of $\mathbf{n}$ is one (a unit vector). Notice the similarity here with the geometric formula for the dot product, which uses $\cos\theta$ as opposed to $\sin\theta$.

The **right-hand rule** is a way of assigning an orientation to the cross product. The idea is to place the pinky of your right hand on $\mathbf{v}$ and then curl your hand toward $\mathbf{w}$. Your thumb then points in the direction of the normal vector $\mathbf{n}$ (see Figure 1).

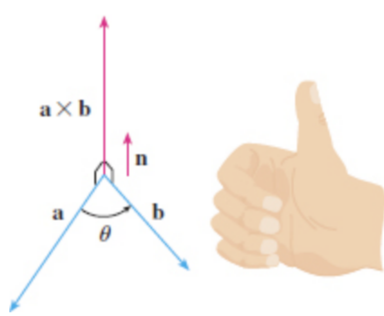

Figure 1: The right-hand rule determines the correct direction of the cross product $\mathbf{a} \times \mathbf{b}$.

Exercise 1: Use the geometric definition of the cross product and the right-hand rule to calculate each of the following:

(a) $\mathbf{i} \times \mathbf{i}$ **(b)** $\mathbf{i} \times \mathbf{j}$ **(c)** $\mathbf{j} \times \mathbf{i}$ **(d)** $(3\,\mathbf{j}) \times (-2\,\mathbf{k})$

Properties of the Cross Product

Suppose that $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ are three-dimensional vectors and that $c \in \mathbb{R}$ is a scalar.

1. $\mathbf{v} \times \mathbf{w}$ is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$, so that $\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w}) = 0$ and $\mathbf{w} \cdot (\mathbf{v} \times \mathbf{w}) = 0$.
2. The vectors $\mathbf{v}$ and $\mathbf{w}$ are parallel if and only if $\mathbf{v} \times \mathbf{w} = 0$.
3. $|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}|\,|\mathbf{w}| \sin\theta$ is equal to the area of the parallelogram formed by $\mathbf{v}$ and $\mathbf{w}$.
4. $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$ (anti-commutative).
5. $\mathbf{v} \times \mathbf{v} = \mathbf{0}$ for any vector $\mathbf{v}$.
6. $(c\,\mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (c\,\mathbf{w}) = c(\mathbf{v} \times \mathbf{w})$ (scalars pull out).
7. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$ and

   $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$ (distributive property).

Some of these properties follow quickly from the definition, while others are a bit more challenging to explain. Property 1 follows directly from the definition of the cross product and the fact that the dot product of two orthogonal vectors is zero. Property 2 follows because parallel vectors either point in the same direction ($\theta = 0$) or the opposite direction ($\theta = \pi$). Since $\sin 0 = \sin \pi = 0$, we see that $\mathbf{v} \times \mathbf{w} = 0$ whenever $\mathbf{v}$ and $\mathbf{w}$ are parallel. This also explains property 5, since $\theta = 0$ in this case.

Property 3 follows from the formula for the area of a parallelogram ($A = bh$) and right-triangle trigonometry (see Figure 2). Property 4 is a consequence of the right-hand rule (order matters!), while property 6 follows from $|c\,\mathbf{v}| = |c|\,|\mathbf{v}|$. The hardest property to justify is property 7, which can be shown using the scalar triple product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ (see Exercise 41 on p. 662 for details).

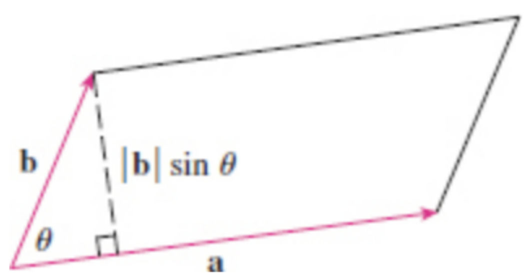

Figure 2: The vectors $\mathbf{a}$ and $\mathbf{b}$ define a parallelogram whose area is $|\mathbf{a}|\,|\mathbf{b}| \sin\theta$, which is equivalent to $|\mathbf{a} \times \mathbf{b}|$.

Exercise 2: Show by way of example that the cross product is **not** associative. In other words, find vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ such that $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \neq \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$.

Algebraic Formula for the Cross Product

Suppose that $\mathbf{v} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{w} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$. We can use the properties of the cross product listed above, along with the identities $\mathbf{i}\times\mathbf{j} = \mathbf{k}$, $\mathbf{j}\times\mathbf{k} = \mathbf{i}$, and $\mathbf{k}\times\mathbf{i} = \mathbf{j}$, to compute the formula for $\mathbf{v}\times\mathbf{w}$. We compute

$$\begin{aligned}
\mathbf{v}\times\mathbf{w} &= (a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}) \times (a_2\,\mathbf{i} + b_2\,\mathbf{j} + c_2\,\mathbf{k}) \\
&= a_1a_2\,\mathbf{i}\times\mathbf{i} + a_1b_2\,\mathbf{i}\times\mathbf{j} + a_1c_2\,\mathbf{i}\times\mathbf{k} \\
&\quad + b_1a_2\,\mathbf{j}\times\mathbf{i} + b_1b_2\,\mathbf{j}\times\mathbf{j} + b_1c_2\,\mathbf{j}\times\mathbf{k} \\
&\quad + c_1a_2\,\mathbf{k}\times\mathbf{i} + c_1b_2\,\mathbf{k}\times\mathbf{j} + c_1c_2\,\mathbf{k}\times\mathbf{k} \\
&= a_1b_2\,\mathbf{k} - a_1c_2\,\mathbf{j} - a_2b_1\,\mathbf{k} + b_1c_2\,\mathbf{i} + a_2c_1\,\mathbf{j} - b_2c_1\,\mathbf{i} \\
&= (b_1c_2 - b_2c_1)\mathbf{i} + (a_2c_1 - a_1c_2)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}\,.
\end{aligned}$$

Thus we have

$$\mathbf{v}\times\mathbf{w} = (b_1c_2 - b_2c_1)\mathbf{i} + (a_2c_1 - a_1c_2)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}\,.$$

This is not a very easy formula to remember; fortunately, there is a simple way to compute the cross product using something called a **determinant**.

The determinant of a 2×2 matrix (array) is the product of one diagonal minus the product of the other. For example,

$$\begin{vmatrix} 4 & 2 \\ -3 & 5 \end{vmatrix} = 4\cdot 5 - 2\cdot(-3) = 20 + 6 = 26.$$

The cross product can be computed by evaluating the determinant of the 3×3 matrix below:

$$\mathbf{v}\times\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}.$$

One method of computing this determinant is to expand it about the first row:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \mathbf{i}\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

Example 1: Compute the cross product of $\mathbf{v} = <1, -4, 2>$ and $\mathbf{w} = <5, 3, -2>$.

Answer: Using the determinant formula, we have

$$\begin{aligned}
\mathbf{v}\times\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 2 \\ 5 & 3 & -2 \end{vmatrix} &= \mathbf{i}\begin{vmatrix} -4 & 2 \\ 3 & -2 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 2 \\ 5 & -2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & -4 \\ 5 & 3 \end{vmatrix} \\
&= \mathbf{i}(8-6) - \mathbf{j}(-2-10) + \mathbf{k}(3+20) \\
&= 2\,\mathbf{i} + 12\,\mathbf{j} + 23\,\mathbf{k}\,.
\end{aligned}$$

Notice that our answer is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$ (check this).

Exercise 3: Let $\mathbf{v} = 3\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}$ and $\mathbf{w} = <1, -1, 2>$. Find $\mathbf{v} \times \mathbf{w}$ and check that it is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.

Exercise 4: Consider the three points $P(1, 0, -2), Q(-1, 3, 1)$, and $R(-2, 2, -1)$ in $\mathbb{R}^3$.

(a) Find a vector orthogonal to the plane containing P, Q, and R.

(b) Find the area of the triangle with vertices P, Q, and R.