MATH 241-02, Multivariable Calculus, Spring 2019

Section 9.3: The Dot Product

How do we multiply two vectors? We have seen that there is a natural geometric and algebraic way to add two vectors. But what about multiplying them together? It turns out that there are *two* different ways to multiply vectors: the **dot product** and the **cross product**. This sections focuses on the dot product.

The Dot Product

Suppose that $\mathbf{v} = <a_1, b_1, c_1>$ and $\mathbf{w} = <a_2, b_2, c_2>$ are two vectors and let θ be the angle between them. To find θ, place the tails of the two vectors at the same point and measure the angle between them, assuming that $0 \leq \theta \leq \pi$. If the vectors point in the same direction, then $\theta = 0$; if they point in the opposite direction, then $\theta = \pi$; and if they are perpendicular, then $\theta = \pi/2$. We give two equivalent definitions of the dot product.

Dot Product:

Algebraic Definition: $\mathbf{v} \cdot \mathbf{w} = a_1a_2 + b_1b_2 + c_1c_2$

Geometric Definition: $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}| \cos\theta$

Thus, the dot product can be found by multiplying corresponding components together and then adding the result. Or, if you know the angle between the two vectors, then the dot product can be found by multiplying the lengths of the two vectors together along with the cosine of the angle between them. If the vectors are two dimensional, then the dot product definition is the same as above except that the c_1c_2 term in the algebraic definition vanishes.

Important Observation: The dot product of two vectors is always a **number** (a scalar)!

Example 1: Compute the dot product of $\mathbf{v} = <1, -4, 0>$ and $\mathbf{w} = <5, 3, -2>$.

Answer: Using the algebraic definition of the dot product we have

$$\mathbf{v} \cdot \mathbf{w} = 1 \cdot 5 + -4 \cdot 3 + 0 \cdot -2 = 5 - 12 + 0 = -7.$$

Note that it is perfectly fine to have a negative value for the dot product. In fact, a negative dot product means the angle between the two vectors is obtuse because $\cos\theta < 0$. Since the lengths of vectors are always positive, it is the $\cos\theta$ term that determines the sign of the dot product.

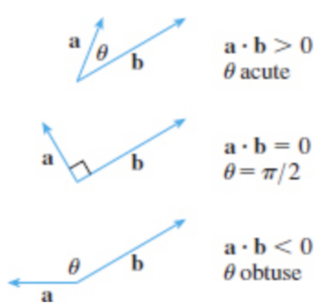

Figure 1: The dot product is positive when the angle between the vectors is acute; it is negative when they are obtuse; and zero if they are perpendicular.

Exercise 1: Let $\mathbf{v} = 2\,\mathbf{i} + 2\,\mathbf{j}$ and $\mathbf{w} = -3\,\mathbf{j}$ be two vectors in the plane. Compute the dot product $\mathbf{v} \cdot \mathbf{w}$ using both the algebraic and geometric definitions and show that they agree.

Exercise 2: Show that the vectors $\mathbf{v} = -\mathbf{i} + 2\,\mathbf{j} + 2\,\mathbf{k}$ and $\mathbf{w} = 2\,\mathbf{i} - 4\,\mathbf{j} + 5\,\mathbf{k}$ are orthogonal (perpendicular).

Exercise 3: Find the angle between the vectors $\mathbf{v} = 2\,\mathbf{i} - 3\,\mathbf{j} + \mathbf{k}$ and $\mathbf{w} = <1, 0, -5>$ to the nearest degree.

Properties of the Dot Product

The following properties of the dot product hold for any vectors **v**, **w**, and **u**. These are straightforward to verify using the algebraic definition of the dot product.

1. $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$
2. $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$ (commutative)
3. $\mathbf{v} \cdot (c\,\mathbf{w}) = (c\,\mathbf{v}) \cdot \mathbf{w} = c(\mathbf{v} \cdot \mathbf{w})$ (scalars pull out)
4. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ and

   $(\mathbf{v} + \mathbf{w}) \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{u}$ (distributive property)

Exercise 4: Let $\mathbf{v} = < a, b, c >$. Show that $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$ using the algebraic definition of the dot product.

Algebraic and Geometric Definitions Match

We now show that the algebraic and geometric definitions of the dot product are equivalent. This can be accomplished using the Law of Cosines. If a triangle has sides of length a, b, and c, and θ is the angle opposite side c, then the Law of Cosines states

$$c^2 \;=\; a^2 + b^2 - 2ab\cos\theta\,.$$

This rule is valid whether θ is acute or obtuse. Let $\mathbf{u}$ and $\mathbf{v}$ be any two vectors, and let θ be the angle between them. Recall that the vector $\mathbf{u}-\mathbf{v}$ forms a triangle with $\mathbf{u}$ and $\mathbf{v}$ (see Figure 2). By the Law of Cosines, we have

$$|\mathbf{u}-\mathbf{v}|^2 \;=\; |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}|\,|\mathbf{v}|\cos\theta\,. \tag{1}$$

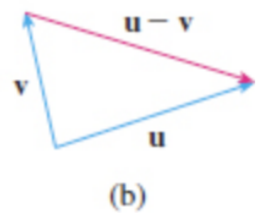

Figure 2: The vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{u}-\mathbf{v}$ form a triangle.

Exercise 5: Using properties of the dot product (all of which follow from the algebraic definition), show that equation (1) simplifies to $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta$. This shows the two definitions are equivalent.

Projections

Our final topic is how to project one vector onto another. This is particularly important in physics and engineering applications as well as many other subjects in mathematics (e.g., geometry). We will denote the **vector projection of b onto a** as $\text{proj}_{\mathbf{a}}\mathbf{b}$. It is obtained by drawing a perpendicular line from the head of $\mathbf{b}$ to $\mathbf{a}$ (see the red vectors in Figure 3). Note that the projection of $\mathbf{b}$ onto $\mathbf{a}$ is a vector pointing in the same direction as $\mathbf{a}$.

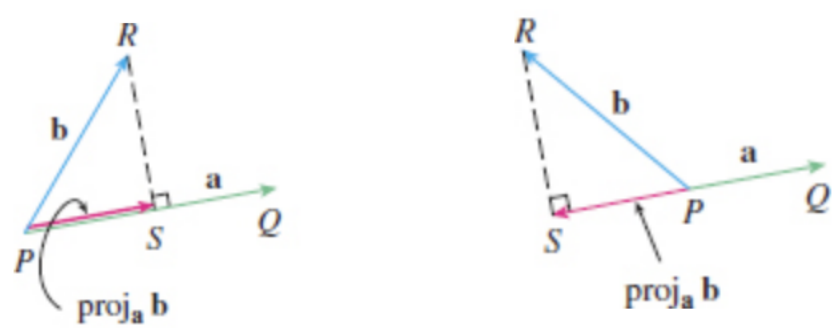

Figure 3: The vector projection of $\mathbf{b}$ onto $\mathbf{a}$, denoted by $\text{proj}_{\mathbf{a}}\mathbf{b}$.

The **scalar projection of b onto a**, also called the **component of b along a**, is the **signed** magnitude of $\text{proj}_{\mathbf{a}}\mathbf{b}$ (the signed length of the red vectors in Figure 3). It is positive if the angle between **a** and **b** is acute and negative if the angle is obtuse. The scalar projection of **b** onto **a** is denoted by $\text{comp}_{\mathbf{a}}\mathbf{b}$.

Important Note: The scalar projection is a **number**, while the vector projection is a **vector**.

To find formulas for $\text{comp}_{\mathbf{a}}\mathbf{b}$ and $\text{proj}_{\mathbf{a}}\mathbf{b}$, we use right-triangle trigonometry. If θ is the angle between **a** and **b**, then the signed length of the scalar projection of **b** onto **a** is $|\mathbf{b}|\cos\theta$ (see Figure 4).

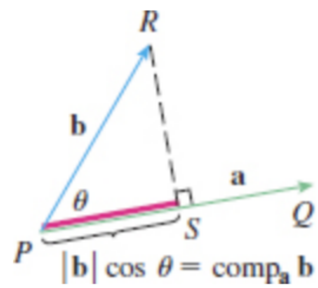

Figure 4: The scalar projection of **b** onto **a**, denoted by $\text{comp}_{\mathbf{a}}\mathbf{b}$, is equal to $|\mathbf{b}|\cos\theta$ since the cosine of an angle in a right triangle equals adjacent/hypotenuse.

Then, using $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$, we see that $|\mathbf{b}|\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$. This gives the formula for $\text{comp}_{\mathbf{a}}\mathbf{b}$.

To obtain the projection of **b** onto **a**, we multiply $\text{comp}_{\mathbf{a}}\mathbf{b}$ by the unit vector in the direction of **a**. Recall that the unit vector in the direction of **a** is given by $\frac{\mathbf{a}}{|\mathbf{a}|}$. This gives the following formulas for the scalar and vector projections.

Scalar projection of **b** onto **a**: $$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$$

Vector projection of **b** onto **a**: $$\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\,\mathbf{a}$$

Exercise 6: Find the scalar and vector projections of $\mathbf{b} = <4, -3, -1>$ onto $\mathbf{a} = <2, 2, -3>$.