Multivariable Calculus, Spring 2019

Computer Project #2

Optimization: Mountains, Celestial Mechanics, and Least Squares

DUE DATE: Friday, April 5, 5:00 pm

This project focuses on two particular applications of multivariable calculus involving optimization. There are many, many problems where extrema of functions are required to find an optimal answer to a real-world question. Here you will rediscover one of the first solutions to the three-body problem in celestial mechanics, and help a field biologist estimate the population of rare toads. You will also compute the maximum height of a mountain. Each problem requires you to find and classify the critical points of a function of two variables. We will make use of the computer software Maple to visualize functions, draw contour diagrams, locate extrema, and simplify our computations.

It is **required** that you work in a group of two or three people. Any help you receive from a source other than your lab partner(s) should be acknowledged in your report. For example, a textbook, web site, another student, etc. should all be appropriately referenced at the end of your report. The project should be typed although you do not have to typeset your mathematical notation. For example, you can leave space for a graph, computations, tables, etc. and then write it in by hand later. You can also include graphs or computations in an appendix at the end of your report. Only **one project per group** should be submitted.

Your report should provide coherent answers to the questions in each section. Be sure to read carefully and answer all of the questions asked. Please do not overload your report (or my attention for reading) by including large numbers of graphs and tables. A well-written report with a few tables and graphs to illustrate key points is far better than a sloppy report with too many figures.

Mountain Peaks

In this "warm-up" exercise, you will use Maple to find the highest peaks of a mountain. Consider the function

$$f(x, y) \ = \ 5 - 10xy - 4x^2 + 3y - y^4,$$

whose output gives the height of a certain mountain (in thousands of feet). The point $(x, y, f(x, y))$ can be thought of as a point on the mountain.

1. Use Maple to find all of the critical points of f, accurate to five decimal places. Some of this problem needs to be done by hand; other parts require Maple. Feel free to use the `plot3d` and `contourplot` commands to help you locate the solutions. The `fsolve` command can be used to solve a system of two equations in two variables with a high degree of accuracy. For example, the command

   ```
   fsolve({x^2 + y^2 = 1, 2*x - y = 0}, {x,y},x=0..1,y=0.3..1);
   ```

   simultaneously solves the system $x^2 + y^2 = 1, y = 2x$ in the region $0 \leq x \leq 1$, $0.3 \leq y \leq 1$. It is important to adjust the region over which you are looking for a solution so that the computer can accurately find an answer. Since there are several extrema, you will need to change the domain in order to locate each critical point.

2. Classify the critical points you found in the previous question as minima, maxima, or saddles. Apply the second derivative test to confirm your findings.

3. Turn in a nice graph of the mountain, indicating the highest points (the local maxima). What is the actual height of each peak?

Celestial Mechanics: The planar circular restricted three-body problem

Consider the following question about the motion of three celestial bodies: Given three masses in space with prescribed initial positions and initial velocities, what is the motion of the bodies over time if the only force acting upon them is their mutual gravitation? Although it was posed centuries ago, we still do not have a complete answer to this question. Concerning the motion of the Earth, Sun, and Moon, Isaac Newton once frustratingly remarked to the astronomer John Machin that "... his head never ached but with his studies on the moon."[1] Although Newton successfully developed his theory of calculus to prove that Kepler's empirical laws were in fact accurate (e.g., the Earth travels on an elliptical orbit with the Sun at one of the focal points), he was unable to find a solution to the motion of three bodies in space, the so-called *three-body problem.* To this day, the three-body problem remains an open question and an active area of research in the fields of mathematics, physics, astronomy, and space transport. The recent *Cassini* mission to Saturn used mathematical information about the three-body problem in order to design low-energy spacecraft trajectories.

One way of approaching this complicated problem is to assume that we know the motion of two massive bodies (say the Earth and the Sun) and then try to ascertain the motion of a third, infinitesimal mass (say the moon or a satellite). Let us assume that the larger bodies, called *primaries*, are on circular orbits about their center of mass and that their motion lies in a plane. The third infinitesimal mass, being so small, does not effect the motion of the larger bodies, whose orbits remain circular. Investigating the motion of the third mass subject to the gravitational attraction from the larger two primaries is called the *planar, circular, restricted, three-body problem.* In this set of exercises, you will find some of the simplest (and earliest) solutions to this particular version of the three-body problem.

Using Newton's law of gravitation, the force between two bodies is proportional to the product of their masses and inversely proportional to the square of the distance between them. Using $F = ma$, we can set up a system of differential equations (a is a second derivative of position) that governs the motion of the infinitesimal particle. We can choose a rotating coordinate frame so that in the new coordinates, the position of the first and second mass appear fixed.

Let μ be a positive real number (a parameter) with $\mu < 1$. Denote $m_1 = \mu$ and $m_2 = 1 - \mu$ as the masses (normalized so that $m_1 + m_2 = 1$) of the two large bodies, and fix their positions to be in the plane at the points $\mathbf{q}_1 = (1 - \mu, 0)$ and $\mathbf{q}_2 = (-\mu, 0)$. This is done so that their center of mass $(m_1 q_1 + m_2 q_2)/(m_1 + m_2)$ is at the origin (check it). Let the position of the third infinitesimal mass be given by (x, y), where each variable is really a function of time. The motion of the third infinitesimal mass involves the *effective potential*, given by

$$V(x,y) \;=\; \frac{1}{2}(x^2+y^2) + \frac{\mu}{\sqrt{(x-1+\mu)^2+y^2}} + \frac{1-\mu}{\sqrt{(x+\mu)^2+y^2}}\;. \tag{1}$$

Note that the denominators of the two fractions in equation (1) give the distances between the large bodies and the small infinitesimal mass. The function V comes from Newton's inverse square law.

It turns out that the critical points (extrema) of the effective potential $V(x, y)$ are *equilibrium solutions* of the differential equations governing the motion of the third body. This is a special solution to the system of differential equations in which the third body is at rest with respect to the other two. The whole system is going around in a circle about the origin so the third body would actually be moving in a circular orbit just like the others. (Think of three points on a spinning record player. Seen from the record, the points are always the same distance apart, while seen from above, they are moving in circles about the center of the record.) These solutions are called *libration points* in classical celestial mechanics literature. They were discovered by the great mathematicians Lagrange and Euler in the mid-18th century and were some of the first solutions found in the three-body problem. They happen to make excellent "parking" spaces for spacecrafts, satellites, or even other celestial bodies such as a moon.

4. Begin by setting $\mu = 1/2$, so $m_1 = m_2 = 1/2$, that is, the primaries have equal mass. One might expect to find some nice symmetry in this case.

   (a) Setting $\mu = 1/2$, what is the domain of $V(x, y)$? Where is $V(x, y)$ undefined and what happens to the effective potential as you approach these points? Note: You could use Maple and the `plot3d` command to view a graph of the function. The `view=-3..3` option is helpful here. Be sure to pick an appropriate range.

   (b) Compute the partial derivatives V_x and V_y BY HAND. Show that $V_x = 0$ at any point on the y-axis and $V_y = 0$ at any point on the x-axis in the domain.

   (c) Find all of the critical points of $V(x, y)$ as accurately as possible and classify them as maxima, minima, or saddles. There are five of them.

   Hint: Use symmetry and the result from the previous question. Follow the same approach you used for the mountain problem. You do **not** need to apply the second derivative test to confirm the types of critical points.

   (d) Draw a graph in the xy-plane containing all the critical points as well as the two positions of the large primaries with masses $m_1 = m_2 = 1/2$. What can you say about the location of each critical point with respect to the two large primaries? Two of the critical points, along with the primaries, form a special shape. Find this shape and use it to determine the *exact* location (no decimals) of these two critical points.

5. Now assume that $\mu = 1/10$, so $m_1 = 1/10$ and $m_2 = 9/10$, that is, the second primary is nine times larger than the first. Change your function $V(x, y)$ accordingly.

   (a) Compute the partial derivatives V_x and V_y BY HAND. Is it still true that $V_x = 0$ at any point on the y-axis and $V_y = 0$ at any point on the x-axis in the domain? Explain.

   (b) Find all of the critical points of $V(x, y)$ as accurately as possible and classify them as maxima, minima, or saddles. There are five of them.

   Hint: Use the result from part **(a)** and follow the same approach you used for the mountain problem. You do **not** need to apply the second derivative test to confirm the types of critical points.

   (c) Draw a graph in the xy-plane containing all the critical points as well as the two positions of the large primaries. Compare your graph to the one obtained in problem #4**(d)**. What is similar and what is different?

Least Squares: How to fit a line to a set of data

Another important optimization problem involves fitting a set of data to a prescribed curve. Data measured in the real world is typically not perfectly linear, quadratic, cubic, exponential, etc. However, it may be well-approximated by a particular line or curve. This exercise gives one way in which to find the "best" such curve, called *the method of least squares*.

Suppose that a biologist working in the field for several months records the following measurements for the number of rare toads, where t is measured in months and n represents the number of toads observed in the hundreds. The data is plotted in Figure 1. The value $n = 20$ at $t = 0$ is an assumption on the part of the biologist based on a previous expedition.

t (months)	n (hundreds)
0	20
0.4	17.3
0.8	18.2
1	17.3
1.5	16.1
2	15.6
2.7	14.1
3	13.5
3.2	12.4
3.8	11.6
4	11
4.1	10.2

Let (t_i, n_i) represent the i-th data point from the table. (So i ranges between 1 and 12.) We want to find the best line such that the sum of the squares of the vertical distance between each point and the line is as small as possible. In other words, suppose that $n = mt + b$ is the equation of the best linear fit to the data. Then the point on the line directly above or below (t_i, n_i) has coordinates $(t_i, mt_i + b)$. We want the distance squared between all such pairs of points to be small, that is, we seek constants m and b such that

$$F(m, b) \;=\; \sum_{i=1}^{12} (n_i - (mt_i + b))^2$$

is as small as possible (a minimum).

6. Plug in the twelve data points above to form the function $F(m, b)$. You can use Maple to do the algebra for you; the hard part is typing in the data correctly. Go slowly! The command `expand( )` will expand and simplify any expression inside the parentheses. For example, `expand( (x-y)^2);` gives the polynomial $x^2 - 2xy + y^2$. You should obtain an expression containing only m and b, a function of two variables! State your simplified version of $F(m, b)$.

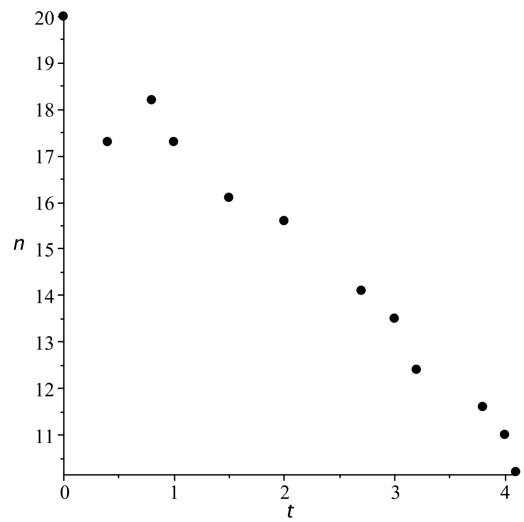

Figure 1: The data collected by a biologist on the number of rare toads n in hundreds versus time t in months.

7. Compute the first partial derivatives of $F(m, b)$ with respect to m and b and find the critical point. This gives the precise value of the slope and n-intercept of the best least squares line. Plot this line on the same graph as the data (use Figure 1). Turn in your graph and state the equation of the line.

8. Compute the second-order partial derivatives and show that the critical point you found in the previous question is indeed a minimum.

9. Based on your linear approximation to the biologist's data, when should we expect the toad species to go extinct?

References

[1] Barrow-Green, J. Poincaré and the Three Body Problem, *History of Mathematics*, vol. 11, American Mathematical Society, pp. 15., 1997.

[2] McCallum, W., Hughes-Hallett, D., et al. Multivariable Calculus, 3rd ed., John Wiley and Sons, Inc., New York, 2002.

[3] Meyer, K. and Hall, G. Introduction to Hamiltonian Dynamical Systems and the N-Body Problem, Springer-Verlag, New York, 1992.