MATH 242: Principles of Analysis, Fall 2009
Logic and Proofs

Statements and Proofs

A *proof* is a sequence of statements starting from one or more assumptions or "hypotheses" and leading by a chain of valid deductions to a "conclusion".

Statements and negation: Let P and Q stand for mathematical *statements*, sentences that are unambiguously true or false. Examples include "$1+1=3$", "$-1<1$", and "for every integer n, $n(n+1)$ is even."

The *negation* of a statement P is the logical opposite of P (sometimes denoted $\sim P$). If P is true, then $\sim P$ is false. Similarly, if P is false, then $\sim P$ is true. In the examples above, the negations are "$1+1\neq 3$", "$-1\geq 1$", and "there exists an integer n such that $n(n+1)$ is not even."

Basic logic: "and" and "or": Statements can be conjoined in two basic ways to yield a third statement. If P and Q are statements, the statement "P or Q" is true exactly when *at least one* of P, Q is true. The statement "P and Q" is true exactly when *both* P and Q are true.

Implications: A sentence of the form "If P, then Q" (or "P *implies* Q") is a *direct implication*. It is often mathematically denoted by $P \Rightarrow Q$. P is called the *hypothesis*, Q the *conclusion*. An implication is *invalid* if P is true but Q is false. Otherwise, the implication is *valid*. This has the peculiar consequence that if P is false and Q is any statement, then "P implies Q" is valid. For example,

> "If $1+1=3$, then the sky is purple,"

is a true statement! The idea is, by starting with a true statement and making valid deductions, you will arrive only at true statements.

Contraposition: The implication "$\sim Q$ implies $\sim P$" is called the *contrapositive* of "P implies Q". An implication and its contrapositive are *logically equivalent*, namely are both valid or both invalid. Practically speaking, you may replace an implication by its contrapositive when proving a theorem or doing a homework problem. You'll often find the contrapositive easier to prove. We'll see some examples as the course progresses.

Converse: The implication "Q implies P" is called the *converse* of "P implies Q". An implication and its converse are quite often *not* logically equivalent. For example,

> "If you are attending Holy Cross, then you are a college student,"

is a true statement whose converse is false.

Logical Equivalence: When an implication $P \Rightarrow Q$ and its converse $Q \Rightarrow P$ are both true, we say that P and Q are *logically equivalent* and write P if and only if Q or P iff Q (mathematically, we write $P \iff Q$.)

All of the above can be captured nicely in a *truth table*.

P	Q	P or Q	P and Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \Leftrightarrow Q$
T	T	T	T	T	T	T
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	F	T	T	T

Example: Let x stand for a real number. Let P be the statement "$x = 1$" and Q be the statement "$x^2 - 1 = 0$". The implication P implies Q is valid. The converse implication, "If $x^2 - 1 = 0$, then $x = 1$" is invalid: The number $x = -1$ is a *counterexample*: It satisfies the converse hypothesis Q, but not the converse conclusion P.

The contrapositive reads, "If $x^2 - 1 \neq 0$, then $x \neq 1$." With a bit of thought you should be able to convince yourself the contrapositive is valid, as it must be on logical grounds.

Principle of Induction: One very important proof technique is called the *Principle of Induction* or simply "proof by induction." Suppose that P_n is a statement that depends on the positive integer n. In other words, as n changes, so does the statement. One example is, "The sum of the numbers in the nth row of Pascal's triangle is 2^n."

Suppose that each of the following statements were true:

1. P_1 is true. (base case)
2. If P_k is true, then P_{k+1} is true. (induction step)

Then, the Principle of Induction states that P_n is true for all $n \in \mathbb{N}$.

Here is a metaphor for induction using an infinite ladder. We want to show that every rung of the ladder can be reached. The base case allows us to start to climb the ladder. The second item being true allows us to climb the next rung of the ladder. Thus, since P_1 is true, the second part of the Principle of Induction allows us to quickly conclude that P_2 is true. But this in turn implies that P_3 is true and so on. Simply stated, if you can get from one rung to the next and you can start at the bottom, then you can climb it all.

To prove a statement using induction, you must show *both* the base case and the induction step are true. We will do a few examples in class.