MATH 242: Principles of Analysis

Homework Assignment #3

DUE DATE: Thurs., Sept. 24, start of class.

Homework should be turned in at the BEGINNING OF CLASS. You should write up solutions neatly to all problems, making sure to show all your work. You are strongly encouraged to work on these problems with other classmates, although the solutions you turn in should be your **own** work. Please cite any references (web based or text) that you may have used for assistance with the assignment.

Note: Please list the names of any students or faculty who you worked with on the top of the assignment.

1. Show that the union of two countable sets is countable. Note: One incorrect proof is to say, "Since both sets are countable, list the elements of one set and after you are finished, list the elements of the other." *Hint:* How did we show that the integers were countable?

2. Use the preceding problem to show that the set of irrational numbers $\mathbf{I}$ is uncountable. Conclude that the set of irrational numbers is much, much "bigger" than the set of rationals.

3. Show that the countable union of countable sets is countable. In other words, suppose that you have a collection of sets A_n, with $n \in \mathbb{N}$, such that A_n is a countable set for each n. Show that the infinite (but countable) union of all the sets A_n is countable. *Hint:* Try to find an explicit method of counting this set which ensures that every element will be counted and counted exactly once.

4. Show that the open intervals $(0, 1)$ and (a, b) have the same cardinality by finding a bijection between them. You may assume that $a < b$.

5. Show that the open interval $(0, 1)$ has the same cardinality as $\mathbb{R}$ by finding a bijection between $(0, 1)$ and $\mathbb{R}$. Based on the previous problem, conclude that the entire real number line and any open interval, no matter how small, have the same size! *Hint:* Use a trig. function.

6. Consider the sequence $x_n = \dfrac{(-1)^n}{2^n}$.

   a) Write out the first six terms of the sequence.

   b) Find the limit of the sequence and prove that it converges to this limit using the ϵ-N defintion of convergence.

7. Find the limit of the sequence $x_n = \dfrac{2 + 5n}{8 + 11n}$ and prove that it converges to this limit using the ϵ-N defintion of convergence.

8. Find the limit of the sequence $x_n = \dfrac{\cos n}{n^2 + 5}$ and prove that it converges to this limit using the ϵ-N defintion of convergence.

9. Do the following exercises from the course text *Understanding Analysis* by Stephen Abbott: **1.4.9, 1.5.4, 2.2.4, 2.2.6**