Mathematical Models

MATH 303 Fall 2018

Midterm Exam Review Solutions

1. **(a)** $1, 3, 4, 7, 11, 18, 29, 47, 76, 123$

   (b) $\lim_{n\to\infty} L_{n+1}/L_n = \phi$ (golden ratio; same argument as the Fibonacci numbers)
   $\lim_{n\to\infty} L_{n+3}/L_n = 2\phi + 1 = \phi^3$. Use

   $$\frac{L_{n+3}}{L_n} = \frac{L_{n+3}}{L_{n+2}} \cdot \frac{L_{n+2}}{L_{n+1}} \cdot \frac{L_{n+1}}{L_n}$$

   and the fact that the limit of the product equals the product of the limits.

   (c) $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} + \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$

   (d) $L_n L_{n+2} - L_{n+1}^2 = 5(-1)^{n+1}$. Proof is similar to problem **2(c)** on HW#1.

2. Self-similarity in the construction of the vascular network of any species (e.g., constant branching ratio, constant ratio between radii of consecutive levels, etc.); the area principle (to ensure a smooth flow through the blood vessels); the volume-preserving principle; capillaries for different animals all have the **same** size and volume (this is probably the most presumptive of their assumptions).

3. Solve $\alpha = 1 + \frac{1}{1+\alpha}$ to find $\alpha = \sqrt{2}$. Solve $\beta = 1 + \frac{1}{2+\beta}$ to find $\beta = \frac{-1+\sqrt{13}}{2}$.

4. **(a)** $7, -1, 1, \frac{1}{2}, \frac{5}{8}, \frac{19}{32}$

   (b) $y_k = \frac{32}{5}\left(-\frac{1}{4}\right)^k + \frac{3}{5}$

   (c) $3/5$

   (d) $3/5$. Solve the equation $y_{k+1} = y_k$.

5. $y_k = \frac{1}{8} \cdot 5^k - \frac{1}{8}(-3)^k$. Sequence is $0, 1, 2, 19, 68, 421, \ldots$

6. **(a)** $1, 2, 3, 4, 5, 6$. Hmmm, is it that simple?

   (b) $y_k = k + 1$. It satisfies the difference equation because

   $$k + 3 = 2(k+2) - (k+1)$$

   is a true statement $\forall\, k$. It also clearly agrees with the initial conditions.

   (c) $y_k = y_0 + (y_1 - y_0)k$. This can be found by realizing that $y_k = c$ is a solution for any constant c. Then, since $y_k = k + 1$ is also a solution, the general solution (by linearity) is $y_k = c + c_1(k+1)$. Now find the correct values of the unknowns c and c_1.

7. **(a)** \$2,417.04. Use the standard formula with $a = 1 + 0.03/12$.

   (b) \$2,417.65. Use the standard formula with $a = (1 + 0.03/365)^{365/12}$. It only increases by roughly 61 cents.

   (c) 13.9 years or 166.856 months. Take the formula

$$P_t = P_0 a^t - M\frac{1 - a^t}{1 - a}, \qquad \text{where } a = e^{0.05/12},$$

   and solve the equation $P_t = 0$ for t. You pay a total of \$46,856.54 in interest.

   (d) It is better to receive \$600 in 5 years if the rate is 3%. To see this, we calculate the present value of \$600 in 5 years, $600e^{-0.03*5} \approx \$516.42$. Since this amount is *greater* than \$500, it is the better deal.

   On the other hand, if the interest rate bumps up to 4%, then it is better to stick with the \$500 today. This follows because the present value of \$600 in 5 years with the new interest rate is $600e^{-0.04*5} \approx \$491.24$. Since this value is *less* than \$500, it is not the better deal. Another way to see this is to compute the value of \$500 compounded continuously for 5 years at the new rate: $500e^{0.04*5} \approx \$610.70$ which is greater than \$600.

   (e) This is a tricky one. We need to compute the present value of each yearly payment and then add them together to see how they compare with \$2 million. The first payment starts the clock (time $t = 0$). We obtain

$$\begin{aligned} 500{,}000 + 500{,}000e^{-0.04\cdot 1} + 500{,}000e^{-0.04\cdot 2} + 500{,}000e^{-0.04\cdot 3} &= \\ 500{,}000\left(1 + e^{-0.04} + e^{-0.08} + e^{-0.12}\right) &\approx \$1{,}885{,}413.11 \end{aligned}$$

   The "loss" on our winnings is 2 million minus the present value or a whopping \$114,586.89.