Mathematical Models MATH 303
Homework Assignment #1
Due Wed., Sept. 5, start of class

You should write up solutions neatly to all problems, making sure to show all your work. A nonempty subset will be graded. You are encouraged to work on these problems with other classmates, and it is ok to use internet sources for help if it's absolutely necessary (with proper citation); however, the solutions you turn in should be your own work and written in your own words.

Note: Please list the names of any students or faculty who you worked with on the assignment.

1. Read Chapter 1, "Fibonacci Numbers, the Golden Ratio, and the Laws of Nature?", from the course textbook *Topics in Mathematical Modeling* by K. K. Tung.

   (a) Find two different flowers somewhere on campus and count the number of petals on each flower. Do you find Fibonacci numbers? List the type of flower and number of petals.

   (b) The author discusses the purported use of the Golden Ratio and Golden Rectangle in art and architecture, e.g., the *Mona Lisa* in Figure 1.7. Use the Internet to find the dimensions of the actual *Mona Lisa*. How well do the proportions agree with the Golden Ratio?

   (c) Go to the Cantor Art Gallery and view the exhibit *SUMMA*, showcasing the work of HC faculty in the Visual Arts. How many pieces can you find that are close to the Golden Ratio? Give names and brief descriptions of each "golden" piece. Be sure not to touch any of the art! Some dimensions are listed on the corresponding placard; others you may need to measure for yourself. Roger Hankins, the Gallery Director, has a tape measure available and has graciously offered to provide assistance to anyone who asks.

2. Recall that the Fibonacci Series (Sequence) is defined by the recursive relation

$$F_{n+2} = F_{n+1} + F_n, \qquad \text{with } F_0 = 1,\ F_1 = 1. \tag{1}$$

   Find explicit formulas for each of the following and then prove them rigorously using induction.

   (a) $F_0 + F_2 + \cdots + F_{2n} =$ ____________.

   (b) $F_0^2 + F_1^2 + \cdots + F_n^2 =$ ____________.

   (c) $F_n F_{n+2} - F_{n+1}^2 =$ ____________.

3. Let $G_n = \dfrac{F_{n+1}}{F_n}$ be the sequence of ratios of consecutive Fibonacci numbers. In this problem we will rigorously prove that the sequence G_n converges to the golden ratio $\phi = (1+\sqrt{5})/2$. You may assume that F_n is an increasing sequence of natural numbers.

   (a) Prove that $\{G_{2n}\}_{n=0}^{\infty}$ is an increasing sequence bounded above. *Hint:* Use your answer to part **(c)** in the previous question.

(b) Check that your proof in part **(a)** can be easily amended to show that $\{G_{2n+1}\}_{n=0}^{\infty}$ is a decreasing sequence bounded below. You don't have to write anything down except for your lower bound.

(c) By the Monotone Convergence Theorem, the sequences G_{2n} and G_{2n+1} each converge, that is,

$$\lim_{n\to\infty} G_{2n} \;=\; L \quad \text{and} \lim_{n\to\infty} G_{2n+1} \;=\; M.$$

Show that $L = M$. *Hint:* Use the property that the limit of the difference of two sequences is equal to the difference of the two limits.

(d) Conclude that $\lim\limits_{n\to\infty} G_n$ exists (call the limit G) and show that $G = \phi$. *Hint:* Divide both sides of the recursive equation in (1) by F_{n+1} and then take the limit of both sides.

4. Show that the continued fraction expansion below is equivalent to the Golden Ratio ϕ:

$$\alpha \;=\; 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}$$

The $\ddots$ indicates to repeat the construction ad infinitum. You may assume that the infinite fraction converges. *Hint:* Notice that the entire expression, call it α, appears within itself (kind of like a self-similar fractal). Write down a finite equation involving α and then solve it.

5. Complete the following exercises from the course textbook:

Chapter 1 (pp. 22–26): # 1, 6 (see the figure below for assistance)

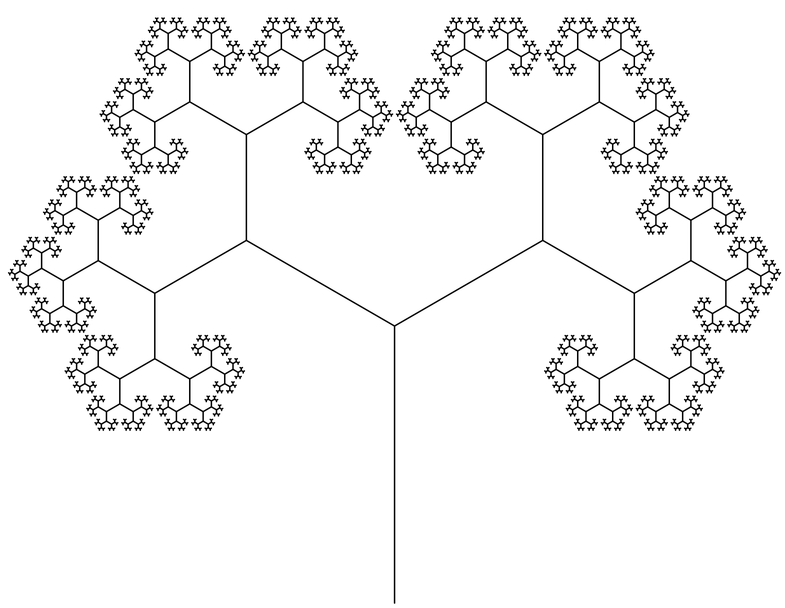

Figure 1: The Golden Tree (Exercise #6). You should explain the presence of the vertical branches. Why does that help you solve the problem?