Mathematical Models   MATH 303

Homework Assignment #2

Due Fri., Sept. 21, 5:00 pm

You should write up solutions neatly to all problems, making sure to show all your work. A nonempty subset will be graded. You are encouraged to work on these problems with other classmates, and it is ok to use internet sources for help if it's absolutely necessary (with proper citation); however, the solutions you turn in should be your own work and written in your own words.

Note: Please list the names of any students or faculty who you worked with on the assignment.

1. Read Chapter 2, "Scaling Laws of Life, the Internet, and Social Networks", from the course textbook *Topics in Mathematical Modeling* by K. K. Tung.

   (a) Following the argument in the text, derive the formula for a finite geometric series with starting term a_0, ratio r, and n terms:

   $$S_n \;=\; a_0 + a_0 r + a_0 r^2 + \cdots + a_0 r^{n-1} \;=\; \frac{a_0(1-r^n)}{1-r}.$$

   (b) What is Zipf's law and how was it discovered? What are some examples where Zipf's law can be found? Describe the work of Wentian Li that somewhat discredits Zipf's law.

2. Complete the following exercises from the course textbook:

   Chapter 2 (pp. 47–53): # 1, 4* (extra credit), 6

   Hints: For #1, use equations (2.9) and (2.10). For #6(a), first find the value of the normalization constant c using the infinite series. For #6(b), use the fact that $\log_M(x) = \ln x / \ln M$ as well as some other standard properties of logs and exponents. To show that $b \approx 1$, consider the area under the graph of $1/x$ from M to $M+1$. If you're game, problem #4 is extra credit.

3. **Solving First- and Second-Order Linear Difference Equations:**

   The recursive equation

   $$y_{k+1} \;=\; a y_k \tag{1}$$

   is an example of a first-order linear difference equation. Given some initial value y_0, iteration of this equation defines a sequence

   $$y_0,\; y_1 = a y_0,\; y_2 = a y_1 = a^2 y_0,\; y_3 = a y_2 = a^3 y_0, \ldots,$$

   and thus by inspection, the solution to the equation is simply $y_k = y_0 a^k$. This formula satisfies equation (1) (plug it into both sides) and it satisfies the initial condition y_0 when $k = 0$.

   Alternatively, we could have guessed the solution has the form $y_k = \lambda^k$. Plugging this into equation (1) and solving for λ gives

   $$\lambda^{k+1} \;=\; a\lambda^k \quad \Longrightarrow \quad \lambda = a,$$

   after dividing through by λ^k. This gives $y_k = a^k$. Because equation (1) is linear, $y_k = c \cdot a^k$ is also a solution for any constant c. This can be checked by plugging into both sides of equation (1). Letting $k = 0$ in $y_k = c \cdot a^k$ gives $y_0 = c \cdot a^0 = c$, so we take $c = y_0$. Once again we obtain the solution $y_k = y_0 a^k$.

(a) Recall from our derivation of Kleiber's Law that the radius r_k and length l_k of a blood vessel at the kth level are each assumed to follow a self-similarity relation:

$$\frac{r_{k+1}}{r_k} = \beta \quad \text{and} \quad \frac{l_{k+1}}{l_k} = \gamma \ \text{ for any } k \in \{0, 1, 2, \ldots, N\},$$

where β and γ are positive constants less than one. Using the formula for the solution to equation (1), explain why

$$\frac{r_k^2 l_k}{r_0^2 l_0} = (\beta^2\gamma)^k.$$

(b) Second-order linear equations can also be solved using the guessing technique. Consider the Fibonacci equation $y_{k+2} = y_{k+1} + y_k$. Substituting in $y_k = \lambda^k$ and solving for λ leads to the quadratic equation $\lambda^2 = \lambda + 1$. Where have we seen this equation before? Let λ_1 and λ_2 be the two solutions of this quadratic equation. We now have two solutions to the difference equation: $y_k = \lambda_1^k$ and $y_k = \lambda_2^k$. In fact, by linearity, the general solution is given by

$$y_k = c_1\lambda_1^k + c_2\lambda_2^k\,,$$

where c_1 and c_2 are arbitrary constants. These constants are determined by the initial values at the start of the sequence, y_0 and y_1. In other words, c_1 and c_2 must satisfy $y_0 = c_1 + c_2$ and $y_1 = c_1\lambda_1 + c_2\lambda_2$ simultaneously. Using this technique, show that the kth Fibonacci number is given by

$$F_k = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{k+1}, \ k \in \{0, 1, 2, \ldots\}.$$

Note that this formula allows for the computation of any Fibonacci number *without* having to know all of its predecessors in the sequence. Surprisingly, despite the presence of the irrational number $\sqrt{5}$, this formula always simplifies to a whole number!

(c) Find the solution to the second-order difference equation

$$y_{k+2} = 3y_{k+1} - 2y_k$$

satisfying the initial conditions $y_0 = 1$ and $y_1 = 3$.

(d) Find the solution to the second-order difference equation

$$y_{k+2} = y_{k+1} + 2y_k$$

satisfying the initial conditions $y_0 = 1$ and $y_1 = 1$.