Fibonacci Numbers and the Golden Ratio: Applications in Nature, Art, and Music

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The Fibonacci Numbers

Definition

The Fibonacci numbers are the numbers in the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots .$$

This is a recursive sequence defined by the equations

$$F_0 = 1, \; F_1 = 1, \quad \text{and} \quad F_{n+2} = F_{n+1} + F_n \quad \text{for all } n \geq 0.$$

Here, F_n represents the n-th Fibonacci number (n is called an index).

Examples: $F_4 = 5, \quad F_9 = 55, \quad F_{102} = F_{101} + F_{100}$.

Often called the “Fibonacci Series” or “Fibonacci Sequence.”

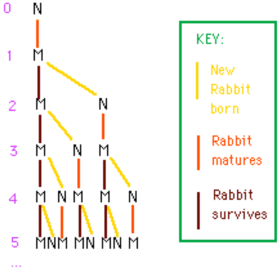
Fibonacci Numbers: History

- Numbers named after Fibonacci by Edouard Lucas, a 19th century French mathematician who studied and generalized them.
- Fibonacci was a pseudonym for Leonardo Pisano (c. 1170–1250). The phrase “filius Bonacci” translates to “son of Bonacci.”
- Fibonacci, an Italian mathematician, was fascinated with computational systems. Wrote important texts reviving ancient mathematical skills.
- Father was a diplomat, so he traveled extensively.
- Imported the Hindu-Arabic decimal system (base 10 as opposed to 60!) to Europe in his book *Liber Abaci* (1202). Latin translation: “Book on Computation.”

Key Passage from the 3rd section of Fibonacci's *Liber Abaci* (1202):

> *"A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?"*

Answer: $233 = F_{12}$. The Fibonacci numbers are generated as a result of solving this problem!



Fibonacci Numbers in Popular Culture

- 13, 3, 2, 21, 1, 1, 8, 5 is part of a code left as a clue by murdered museum curator Jacque Saunière in Dan Brown's best-seller *The Da Vinci Code*.

- Crime-fighting FBI math genius Charlie Eppes mentions how the Fibonacci numbers occur in the structure of crystals and in spiral galaxies in the Season 1 episode "Sabotage" (2005) of the television crime drama *NUMB3RS*.

- Fibonacci numbers feature prominently in the Fox TV Series *Touch* (2012–13), concerning a mathematically gifted boy who is mute but strives to communicate to the world through numbers.

  > *Patterns are hidden in plain sight, you just have to know where to look. Things most people see as chaos actually follows subtle laws of behaviour; galaxies, plants, sea shells.* – from the opening narration

- The hip hop duo *Black Star* repeats the following lyrics in "Astronomy (8th Light):"

  Now everybody hop on the one, the sounds of the two
  It's the third eye vision, five side dimension
  The 8th Light, is gonna shine bright tonight

- In the song "Lateralus," by the American rock band *Tool*, the syllables in the verses (counting between pauses) form the sequence

$$1, 1, 2, 3, 5, 8, 5, 3, 2, 1, 1, 2, 3, 5, 8, 13, 8, 5, 3.$$

  The time signatures vary between $\frac{9}{8}$, $\frac{8}{8}$, and $\frac{7}{8}$, so the song was originally titled "987."
  $F_{15} = 987$, a Fibonacci number!

Fibonacci Numbers in the Comics

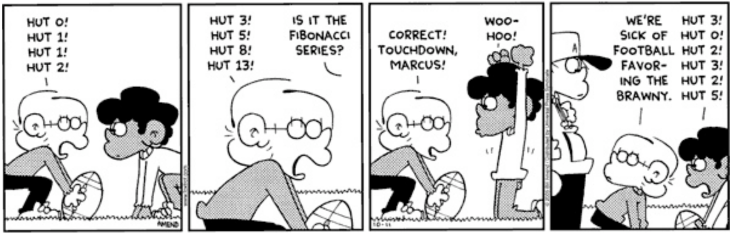


Figure: *FoxTrot* by Bill Amend (2005)

Fibonacci Numbers in Art

Figure: The chimney of Turku Energia in Turku, Finland, featuring the Fibonacci sequence in 2m high neon lights (Mario Merz, 1994).

Figure: Fibonacci Cubes (Petra Paffenholz, 2014), a sculpture situated on two meadows near Lake Dümmer (Germany), consists of nine iron cubes with dimensions based on the Fibonacci numbers.

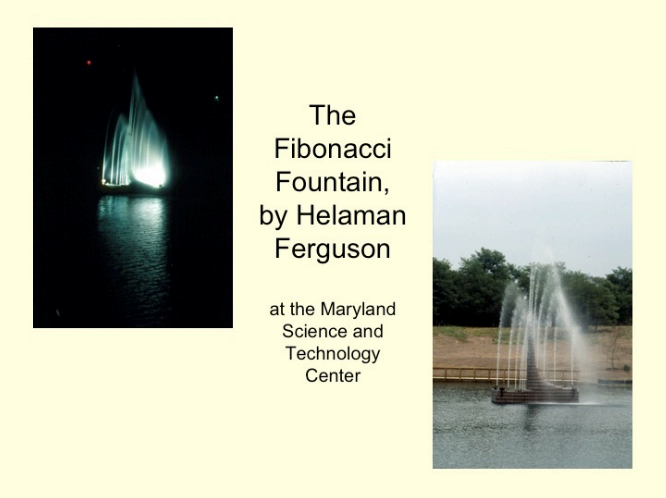


Figure: The fountain consists of 14 (?) water cannons located along the length of the fountain at intervals proportional to the Fibonacci numbers. It rests in Lake Fibonacci (reservoir).

Fibonacci Poetry

"Heart Symphony," a Fib by the poet Silent One

My
soul
sings a
symphony
of perpetual
omniscient narrative lyrics.
Tones reminiscent of azure bluebird lullabies.
Enchanting like stars in indigo skies and blossoming like fragile fragrant bluebells

The number of syllables in each successive line: 1, 1, 2, 3, 5, 8, 13, 21.

Rhythmic Patterns in Sanskrit Poetry

- Indian scholars such as Gopala (c. 1135) and Hemachandra (1089–1173) studied cadences in Sanskrit poetry. Syllables are either short (S), lasting one beat, or long (L), lasting two beats.

- Natural question: How many ways is it possible to divide n beats into short ($S = 1$) and long ($L = 2$) beats?

n	**patterns**	**# of possible divisions**
1	S	1
2	S S, L	2
3	S S S, L S, S L	3
4	S S S S, L S S, S L S S S L, L L	5
5	S S S S S, L S S S, S L S S S S L S, L L S S S S L, L S L, S L L	8

Rhythmic Patterns in Sanskrit Poetry (cont.)

- A recursive pattern emerges. The number of possible divisions for the next n-value is equal to the sum of the possibilities of the previous two values.

- If H_n equals the number of ways to subdivide n beats into S-L patterns, then
$$H_n = H_{n-1} + H_{n-2}.$$
For example, $H_6 = H_5 + H_4 = 8 + 5 = 13$.

- The resulting sequence of numbers, $1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$, are called the Hemachandra numbers.

- Hemachandra's "proof": Every n-beat rhythm ends with either an L or an S. That's it!

- To find the number of ways to subdivide n beats, take all the possibilities for $n - 1$ beats and append an S, and take those for $n - 2$ and append an L.

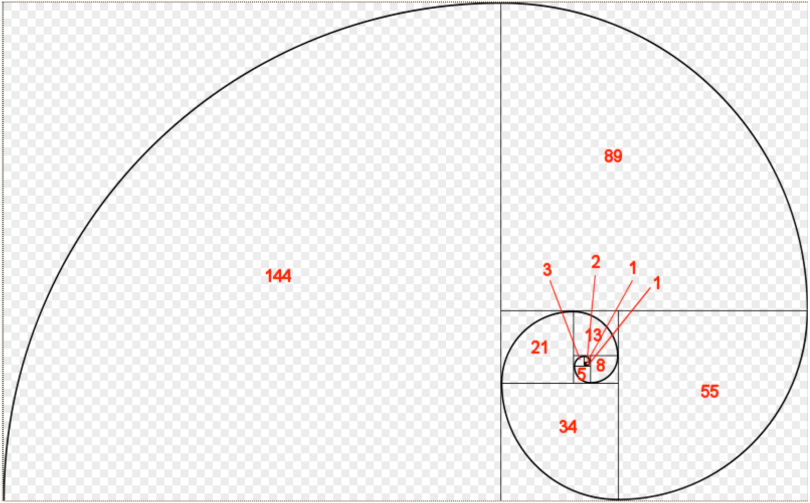


Figure: The Fibonacci Spiral, an example of a Logarithmic Spiral, very common in nature.

Chambered nautilus shell

Figure: The Pinwheel Galaxy (also known as Messier 101 or NGC 5457).

A Fibonacci Identity

Consider the sum of consecutive Fibonacci numbers:

$$F_0 + F_1 + F_2 + F_3 + \cdots + F_n$$

Class: Try a few cases with different n-values and then find an explicit formula for the sum (e.g., $F_n^2 + 3$). Prove your formula using induction.

Fibonacci Numbers in Nature

- Number of petals in many, many flowers: e.g., three-leaf clover, buttercups (5), black-eyed susan (13), daisies (21 or 34).
- Number of spirals in bracts of a pine cone or pineapple, in both directions, are typically consecutive Fibonacci numbers.
- Number of leaves in one full turn around the stem of some plants.
- Number of spirals in the seed heads on daisy and sunflower plants.
- This is not a coincidence! Some of these facts can be modeled mathematically using continued fractions and the golden ratio.

Figure: My three-year old research assistants, Owen (now 9) and Vivian, counting flower petals.

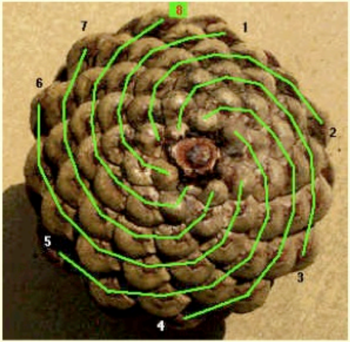
 

Figure: Red Columbine (left, 5 petals, source: ljhimages/iStock/Thinkstock); Black-eyed Susan (right, 13 petals, source: herreid/iStock/Thinkstock)

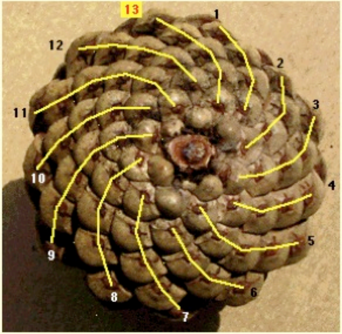
 

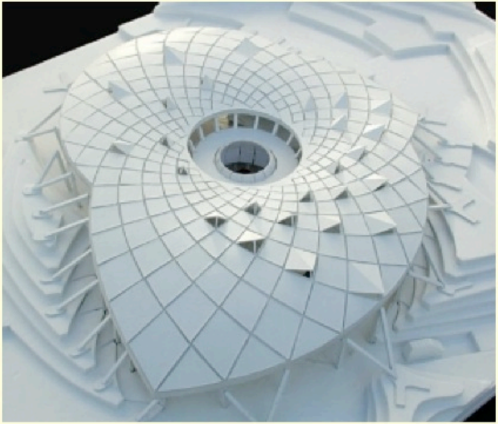
Figure: Chicory (left, 21 petals, source: ArminStautBerlin/iStock/Thinkstock); Sunflower (right, 34 petals, source: Racide/iStock/Thinkstock)

Bracts arranged in Fibonacci numbers of spirals



Adjacent Fibonacci numbers, 8, 13





Model of the Education Building, The Eden Project, Cornwall
by Joylon Brewis and Peter Randall-Page

192 Limits

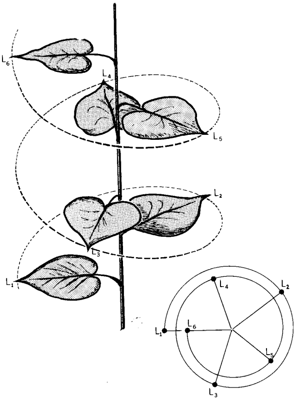
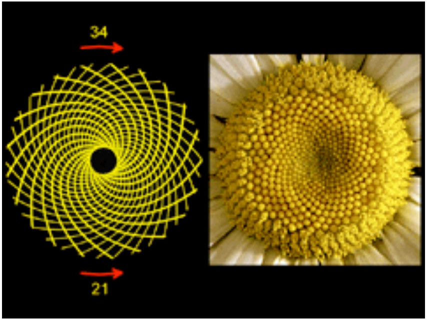


Fig. 8.8. Helical arrangement of leaves on a stem. In the figure it is assumed that the same pattern with five leaves is repeated after two full windings of the helix. This is the case in roses, some willows and cherries. Left: view from the side. Right: View from top

Figure: Excerpt from the text “Introduction to Mathematics for Life Sciences,” by Edward Batschelet (1971), demonstrating the occurrence of Fibonacci in the number of leaves (5) and windings (2) per “period” (when the same leaf orientation returns). Botanists would say the Phyllotactic ratio is $2/5$.



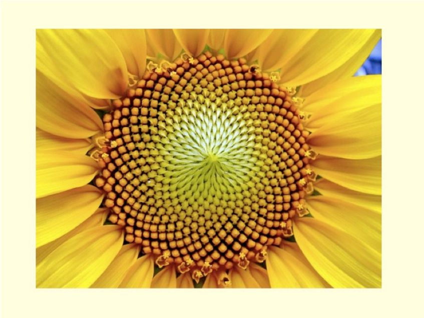


Figure: In most daisy or sunflower blossoms, the number of seeds in spirals of opposite direction are consecutive Fibonacci numbers.

Plant Spirals: Beauty You Can Count On — exhibit at the Botanical Garden of Smith College (2002–03)

How can mathematics help explain the prevalence of Fibonacci numbers in nature?

The Golden Ratio

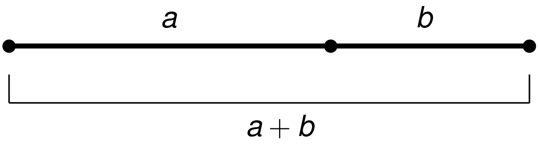


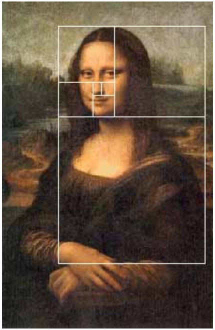
Figure: The ratio $a : b$ equals the ratio $(a + b) : a$, called the Golden Ratio.

$$\frac{a+b}{a} = \frac{a}{b} \implies \phi = \frac{a}{b} = \frac{1+\sqrt{5}}{2} \approx 1.61803398875$$

Class: Find an equation in terms of ϕ. Solve it to find the value listed above.

The Golden Ratio in Art and Architecture

The Golden Ratio ϕ, also known as the Golden Mean, the Golden Section and the Divine Proportion, is thought by many to be the most aesthetically pleasing ratio (e.g., Leonardo da Vinci is purported to have used it in the “Mona Lisa.”)



It was known to the ancient Greeks (e.g., Pythagoras and Euclid) and its use has been speculated in their architecture and sculptures (e.g., the Parthenon).

The Pentagram

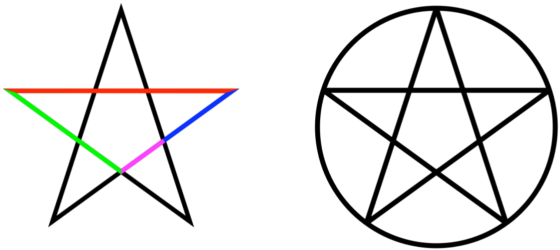


Figure: Left: the Pentagram — each colored line segment is in a Golden Ratio proportion to the next smaller colored line segment. Right: the Pentacle (a pentagram inscribed inside a circle.)

The Pythagoreans used the Pentagram (they called it *Hugieia*, "health") as their symbol in part due to the prevalence of the golden ratio in the line segments.

The pentagram is a five-pointed star that can be inscribed in a circle with equally spaced vertices (a regular pentagon).

Fibonacci Numbers and The Golden Ratio

Consider the ratios of successive Fibonacci numbers:

$$\frac{1}{1} = 1, \quad \frac{2}{1} = 2, \quad \frac{3}{2} = 1.5, \quad \frac{5}{3} = 1.66\bar{6}, \quad \frac{8}{5} = 1.6, \quad \frac{13}{8} = 1.625,$$

$$\frac{21}{13} \approx 1.6154, \quad \frac{34}{21} \approx 1.6190, \quad \frac{55}{34} \approx 1.6176, \quad \frac{89}{55} \approx 1.6182,$$

$$\frac{144}{89} \approx 1.617978, \quad \frac{233}{144} \approx 1.618056, \quad \frac{377}{233} \approx 1.618026$$

Recall: $\phi \approx 1.61803398875$

Fibonacci Fun Fact #1: (HW)

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi, \quad \text{the Golden Ratio!}$$

An Important Observation

This fact does not depend on the opening numbers of the sequence.

In other words, if you create a sequence using the recursive relation $G_{n+2} = G_{n+1} + G_n$, then regardless of your starting numbers, the limit of the ratio of successive terms will be the Golden Ratio.

Example: 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, ...

$$\frac{843}{521} \approx 1.618042$$

These numbers will be important later: they are called the Lucas Numbers.

Definition

Given a real number α, the continued fraction expansion of α is

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}} = [a_0; a_1, a_2, a_3, \ldots]$$

where each a_i (except possibly a_0) is a positive integer.

Example:

$$\alpha = 3 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}} = [3; 2, 4, 1, \ldots]$$

More on Continued Fractions

Question: How do we find α if it is an infinite fraction?

Answer: One approach is to approximate α by terminating the fraction at different places. These approximations are called convergents. In general, the

$$n^{\text{th}}\text{convergent to } \alpha \;=\; \frac{p_n}{q_n} \;=\; [a_0; a_1, a_2, a_3, \ldots, a_n] \, .$$

The larger n is (the further out in the expansion we go), the better the approximation to α becomes.

Key Idea: Any particular convergent p_n/q_n is closer to α than any other fraction whose denominator is less than q_n. The convergents in a continued fraction expansion of α are the best rational approximations to α.

An Important Example

Consider $\alpha = [1; 1, 1, 1, \ldots]$. The first five convergents are:

$$\frac{p_0}{q_0} = 1 = \frac{1}{1}, \qquad \frac{p_1}{q_1} = 1 + \frac{1}{1} = \frac{2}{1},$$

$$\frac{p_2}{q_2} = 1 + \cfrac{1}{1 + \cfrac{1}{1}} = \frac{3}{2},$$

$$\frac{p_3}{q_3} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}} = \frac{5}{3},$$

$$\frac{p_4}{q_4} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}} = \frac{8}{5}.$$

An Important Example (cont.)

We've seen this before:

$$\lim_{n\to\infty} \frac{p_n}{q_n} = \phi, \quad \text{the Golden Ratio!}$$

In other words, $\alpha = [1; 1, 1, 1, \ldots] = \phi$. (HW)

Fibonacci Fun Fact #2: The convergents in the continued fraction expansion of the Golden Ratio ϕ are the ratios of successive Fibonacci numbers.

This means that the Fibonacci fractions give the best approximations to the Golden Ratio.

Fibonacci Phyllotaxis

In 1994, botanist Roger Jean conducted a survey of the literature encompassing 650 species and 12,500 specimens. He estimated that among plants displaying spiral or multijugate phyllotaxis ("leaf arrangement") about 92% of them have Fibonacci phyllotaxis. (*Phyllotaxis: A Systemic Study in Plant Morphogenesis*, Cambridge University Press, 1994)

Question: Why do so many plants and flowers feature Fibonacci numbers?

Succint Answer: Nature tries to optimize the number of seeds in the head of a flower. Starting at the center, each successive seed occurs at a particular angle to the previous, on a circle slightly larger in radius than the previous one. This angle needs to be an irrational multiple of 2π, otherwise there is wasted space. But it also needs to be poorly approximated by rationals, otherwise there is still wasted space.

Fibonacci Phyllotaxis (cont.)

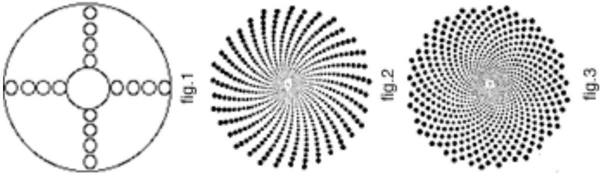


Figure: Seed growth based on different angles β of dispersion. Left: $\beta = 90^\circ$. Center $\beta = 137.6^\circ$. Right: $\beta = 137.5^\circ$.

Question: What is so special about 137.5°? It's the golden angle!

Dividing a circle ($360^\circ = 2\pi$) by the Golden Ratio ϕ, and choosing the lesser angle gives

$$\beta \;=\; \pi(3-\sqrt{5}) \;\approx\; 137.5077641^\circ.$$

This seems to be the best angle available.

Example: The Golden Angle

Figure: The Aonium with 3 CW spirals and 2 CCW spirals. Below: The angle between leaves 2 and 3 and between leaves 5 and 6 is very close to 137.5°.

Why ϕ?

- Recall: $\phi = \frac{1+\sqrt{5}}{2}$ is an irrational number. Moreover, the continued fraction expansion of ϕ is $[1; 1, 1, 1, \ldots]$. Because the terms in the continued fraction are all 1 (no growth in the a_i's), the least "rational-like" irrational number is ϕ.

- On the other hand, the convergents (the best rational approximations to ϕ) are ratios of successive Fibonacci numbers.

- Since an approximation must be made (the number of seeds or leaves are whole numbers), Fibonacci numbers are the best choice available. Since the petals of flowers are formed at the extremities of the seed spirals, we also see Fibonacci numbers in the number of flower petals too!

Wow! Mother Nature Knows Math.

Fibonacci in Music? The Bartók Controversy

- Béla Bartók: Born in Nagyszentmiklós, Hungary (now Sînnicolau Mare, Romania) in 1881. Died in New York, Sept. 1945.
- Studied at the Catholic Gymnasium (high school) in Pozsony where he excelled in math, physics, and music. Enters the Academy of Music (Liszt was 1st president) in Budapest in 1899.
- Avid collector of folk music (particularly Hungarian, Romanian, Slovakian and Turkish).
- Influenced by Debussy and Ravel; preferred Bach to Beethoven.
- Considered to be one of Hungary's greatest composers.

Figure: Bartók at age 22.

- Very interested in nature. Builds impressive collection of plants, insects, and minerals. Fond of sunflowers and fir-cones.

- “We follow nature in composition ... folk music is a phenomenon of nature. Its formations developed as spontaneously as other living natural organisms: the flowers, animals, etc.” — Bartók, *At the Sources of Folk Music* (1925)

- Notoriously silent about his own compositions. “Let my music speak for itself, I lay no claim to any explanation of my works!”

Ernö Lendvai

- In 1955, the Hungarian musical analyst Ernö Lendvai started to publish works claiming the existence of the Fibonacci numbers and the Golden Ratio in many of Bartók's pieces.

- Some find Lendvai's work fascinating and build from his initial ideas; others find errors in his analysis and begin to discredit him. Lendvai becomes a controversial figure in the study of Bartók's music.

- Lendvai draws connections between Bartók's love of nature and "organic" folk music, with his compositional traits. He takes a broad view, examining form (structure of pieces, where climaxes occur, phrasing, etc.) as well as tonality (modes and intervals), in discerning a substantial use of the Golden Ratio and the Fibonacci numbers.

Example: *Music for Strings, Percussion and Celesta*, Movement I

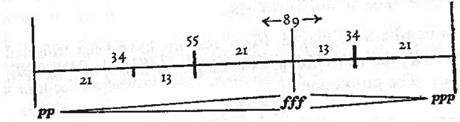


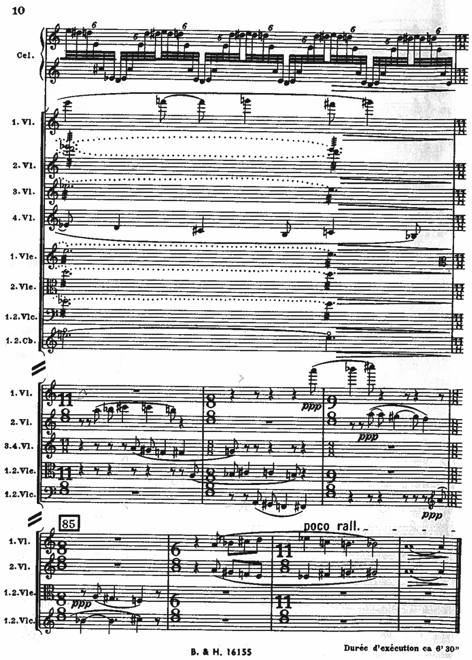
FIG. 22

Lendvai's analysis states:

1. Piece is 89 measures long.
2. The climax of the movement occurs at the end of bar 55 (loudest moment), which gives a subdivision of two Fibonacci numbers (34 and 55) that are an excellent approximation to the golden ratio.
3. String mutes are removed in measure 34.
4. The exposition in the opening ends after 21 bars.

Voilà: Fibonacci and the Golden Ratio


1
Musik für Saiteninstrumente,
Schlagzeug und Celesta (in 4 Sätzen)
Musique pour instruments à cordes,
percussion et célesta (en 4 parties)
I.
Andante tranquillo, ♪ ca 116-112
Béla Bartók
con sord.
1. 2. Viole
pp
5
10
15
Copyright 1937 by Universal Edition.
Copyright assigned 1939 to Boosey & Hawkes, Ltd.,
for the British Empire, North, Central and South America.
B. & H. 16155


10
Cel.
1. Vl.
2. Vl.
3. Vl.
4. Vl.
1. Vle.
2. Vle.
1.2. Vlc.
1.2. Cb.
3.4. Vl.
1.2. Vle.
85
poco rall.
ppp
B. & H. 16155
Durée d'exécution ca 6' 30"

Problems with Lendvai's Analysis (Roy Howat)

1. The piece is 88 bars long, not 89! Lendvai includes a footnote: "The 88 bars of the score must be completed by a whole-bar rest, in accordance with the Bülow analyses of Beethoven." Hanh?!

2. The dynamic climax (*fff*) of the piece is certainly at the end of measure 55. But the tonal climax is really at bar 44, when the subject returns a tritone away from the opening A to E$\flat$. $(88/2 = 44$, symmetry?)

3. The viola mutes come off at the end of bar 33 (not 34) while the first violins and cellos remove their mutes at the start of measure 35 (again, not 34). Only the second, third, and fourth violins remove their mutes in bar 34.

4. The fugal exposition actually ends in bar 20, not 21.

5. What about the celesta? It enters after bar 77. Not even close to a Fibonacci number!

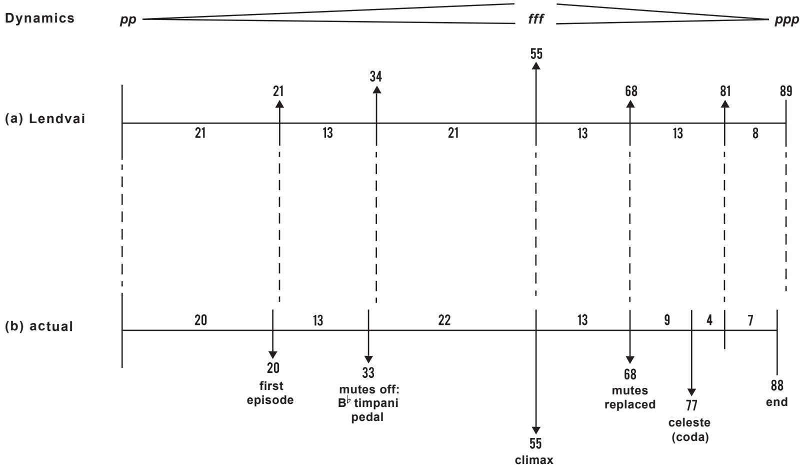
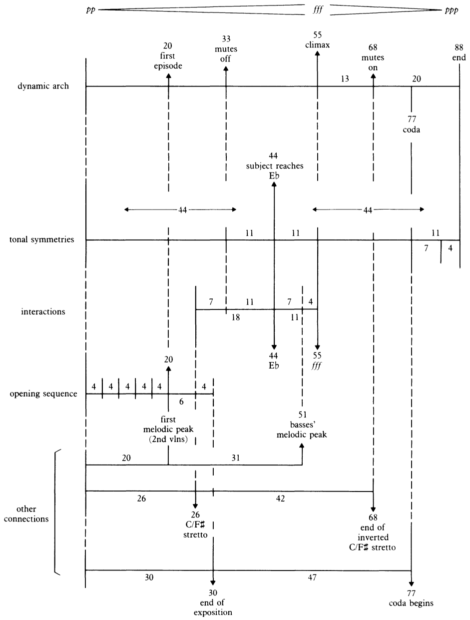
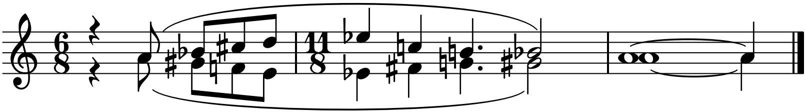


Figure: Roy Howat's analysis of Lendvai's work, from "Bartók, Lendvai and the Principles of Proportional Analysis," *Music Analysis*, **2**, No. 1 (March, 1983), pp. 69-95.

Fig. 5: Fugue from Music for Strings, Percussion and Celeste

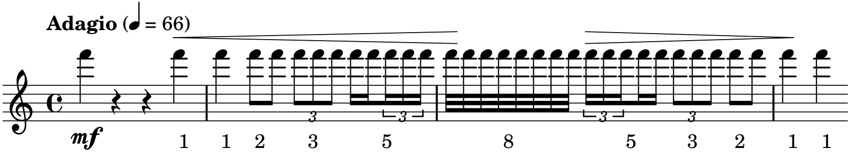


A dramatic and revealing exact inversion based on the scale of the main theme occurs at the end of the first movement.



- Top part is first violins; bottom part is second violins. All other instruments are silent.
- The inversion is about A, reaffirming it as the tonal center of the movement. The motion from A to E♭ and back to A recaps the tonal structure of the fugue. Key idea: symmetry.
- Who was the master of using inversions in fugues? Bach!
- Last four notes: C B♮ B♭ A, which translates in German (Bach's native tongue) to C H B A. Coincidence?

Music for Strings, Percussion and Celesta, Movement III



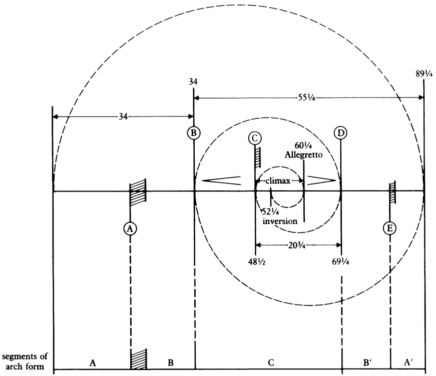
- Opening xylophone solo has the rhythmic pattern

$$1, 1, 2, 3, 5, 8, 5, 3, 2, 1, 1,$$

  with a crescendo followed by a decrescendo (hairpin) climaxing at the top of the sequence. Obvious nod to Fibonacci as well as a nice use of retrograde symmetry.

- Music for this movement famously used by Stanley Kubrick in his film adaptation of Stephen King's *The Shining*.

Fig 7: third movement of Music for Strings, Percussion and Celeste



Howat's analysis of the third movement suggests a greater connection to the Fibonacci numbers and the golden ratio than in the first movement. When counting by the number of quarter notes (assuming $\frac{4}{4}$ time), the piece has 89 measures and a major subdivision into 34- and 55-measure sections.

Ex. 2: Facsimile of recto pages 1 and 2 from manuscript 80FSS1 in the New York Béla Bartók Archive, reproduced by kind permission of Dr Benjamin Suchoff, Trustee of the Bartók Estate.

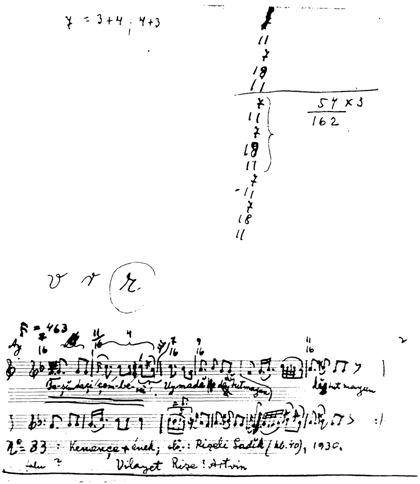


Figure: If you dig deep enough ... Bartók's analysis of a Turkish folk song showing the Lucas numbers!

Some Final Remarks on the Bartók Controversy

- Lendvai's inaccuracies partly due to a narrow focus on the Fibonacci numbers. It appears that the Lucas numbers were just as significant in the first movement of *Music for Strings, Percussion and Celesta*.
- Strength of first movement lies in its use of symmetry:
  1. Tonal climax in measure 44, half way through piece.
  2. Inverting the subject exactly after the climax in measure 55.
  3. Tonal symmetry built around A; mirrored trip around circle of fifths.
  4. Wonderful exact inversion at the end of the piece.

Final Remarks (cont.)

- Other works by Bartók where the golden ratio can be detected are *Sonata for Two Pianos and Percussion*, *Miraculous Mandarin*, and *Divertimento*.
- Bartók was highly secretive about his works. Surviving manuscripts of many of the pieces where the golden ratio appears to have been used contain no mention of it.
- Bartók was already being criticized for being too "cerebral" in his music. Identifying the mathematical patterns in structure and tonality (even to his students!) would only have added fuel to the fire.
- Bottom line: Plenty of evidence in support of mathematical ideas at work in *Music for Strings, Percussion and Celesta*, but don't fudge the analysis!

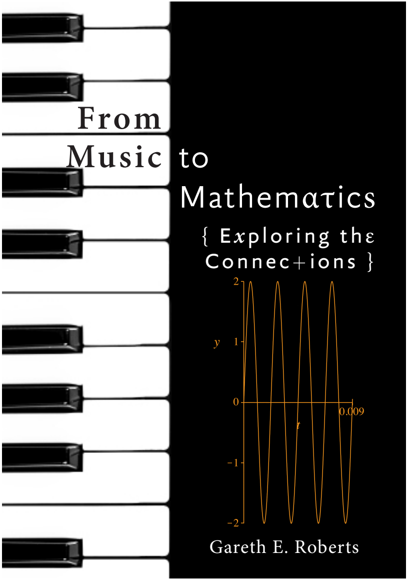


Figure: Textbook (Johns Hopkins University Press, 2016).