MATH 374, Dynamical Systems, Fall 2017
Computer Project #1: Rates of Convergence
Estimates for Neutral Fixed Points and Cycles

Recall: For an attracting fixed point p with $|f'(p)| = k < 1$, we showed in the proof of the Attracting Fixed Point Theorem (AFPT) that

$$|f^n(x) - p| \;<\; k^n|x - p|$$

for x-values sufficiently close to p. This means that the distance between the nth iterate and the fixed point p decreases exponentially at a rate of k^n. Thus, the closer k is to 0, the faster the rate of convergence. This explains why super-attracting fixed points ($k = 0$) attract nearby points the fastest—hence the name "super-attracting."

The Neutral Case: $|f'(p)| = 1$

Slow convergence occurs when $k = 1$, since $k^n = 1$ for any value of n. Here, the AFPT does not apply, although graphical analysis (i.e., web diagram) indicates weak convergence.

Exercise 4: $f(x) = x^2 + 0.25, p = 1/2, f'(p) = 1$

For this example, p weakly attracts initial seeds to the left (such as $x_0 = 0.2$). To estimate the rate of convergence, we use the Taylor series of f about $p = 1/2$:

$$f(x) \;=\; x^2 + \frac{1}{4} \;=\; \frac{1}{2} + \left(x - \frac{1}{2}\right) + \left(x - \frac{1}{2}\right)^2. \tag{1}$$

The graph of f has a quadratic tangency to the diagonal $y = x$ at $x = 1/2$.

Let x be a point slightly less than 1/2 and let $\delta = |x - 1/2| = 1/2 - x$ represent the distance between x and the fixed point. Then, using equation (1), we have $1/2 - f(x) = 1/2 - x - (1/2 - x)^2$ or

$$\left|\frac{1}{2} - f(x)\right| \;=\; \delta - \delta^2. \tag{2}$$

Equation (2) means that $f(x)$ has moved only δ^2 toward the fixed point, not $k\cdot\delta$ as with an attracting fixed point. If we apply this argument again, we see that $f^2(x)$ will be $(\delta - \delta^2)^2 = \delta^2 - 2\delta^3 + \delta^4$ away from $p = 1/2$. Ignoring the higher order terms (which become less and less important as $\delta \to 0$), we see that each iterate is approximately δ^2 closer to the fixed point than before. The reason the convergence is so slow is because the closer we get to the fixed point, the smaller δ, and hence δ^2, becomes. Thus, even though we move closer to p, the amount we move toward it becomes less and less the closer we get.

Based on the above calculations, a good estimate for the number of iterates needed to move within ϵ of the fixed point comes from the equation

$$n \cdot \epsilon^2 \approx \epsilon \qquad \Longrightarrow \qquad n \approx \frac{1}{\epsilon}.$$

This is a remarkably good estimate for small ϵ. For instance, if $\epsilon = 0.00001$, we get $n \approx 100{,}000$, which is very close to the actual answer of 99,987.

Exercise 5: $f(x) = x^2 - 0.75, p = -1/2, f'(p) = -1$

As observed by all lab groups, this example, which also involves a neutral fixed point, takes exceedingly long to converge within ϵ of the fixed point. Setting the digits command to 25 in Maple gives $n = 4{,}999{,}949{,}965$ iterates, a calculation that took overnight to finish computing! Why is this example so much slower to converge than the previous one?

For starters, the slope at the fixed point is negative. This means the orbit **oscillates** about the fixed point as it approaches it. The key is to consider the second iterate,

$$f^2(x) \;=\; \left(x^2 - \frac{3}{4}\right)^2 - \frac{3}{4} \;=\; x^4 - \frac{3}{2}x^2 - \frac{3}{16}.$$

Computing a Taylor series expansion of $f^2(x)$ about $p = -1/2$ gives

$$f^2(x) \;=\; -\frac{1}{2} + \left(x + \frac{1}{2}\right) - 2\left(x + \frac{1}{2}\right)^3 + \left(x + \frac{1}{2}\right)^4.$$

Note that there is no quadratic term in this expansion, because $(f^2)''(-1/2) = 0$. In this case, the graph of f^2 has a **cubic** tangency to the diagonal $y = x$, rather than a quadratic tangency as in the previous problem. This accounts for the slower rate of convergence. If we let $\delta = |x - (-1/2)| = x + 1/2$ (since $x > -1/2$), we find that

$$|f^2(x) - p| \;=\; |f^2(x) + \frac{1}{2}| \;=\; \delta - 2\delta^3 + \delta^4.$$

Thus, one iterate of $f^2(x)$ moves us roughly $2\delta^3$ closer to the fixed point p as opposed to δ^2 (a larger number) in the previous example.

As with the previous exercise, a good estimate for the number of iterates needed to move within ϵ of the fixed point comes from the equation

$$n \cdot 2\epsilon^3 \approx \epsilon \qquad \Longrightarrow \qquad n \approx \frac{1}{2\epsilon^2}.$$

This is a remarkably good estimate for small ϵ. For instance, if $\epsilon = 0.00001$, we get $n \approx 5$ billion, which is very close to the actual answer.

Exercise 14: $f(x) = x^2 - 1.25, q_\pm = (-1 \pm \sqrt{2})/2, (f^2)'(q_\pm) = -1$

This problem is similar to the last one, except now we focus on the fourth iterate f^4. Here the convergence is also relatively slow due to a cubic tangency. We find it takes $n = 292{,}871{,}501$ iterates to get within $\epsilon = 0.00001$ of q_- and $n = 1{,}644{,}065{,}030$ to move within ϵ of q_+. The discrepancy in convergence between the two periodic points comes from the fact that the third derivative of f^4 is different (by a factor of around 5.8) at each point, which then leads to different Taylor series expansions.

As with the previous problems, Taylor series calculations show that the one iterate of f^4 moves us $(40 + 20\sqrt{2})\delta^3$ closer to the periodic point q_- and $(40 - 20\sqrt{2})\delta^3$ closer to q_+. This in turn leads to the estimates

$$n \;\approx\; \frac{2}{(40 + 20\sqrt{2})\epsilon^2} \;\approx\; 292{,}893{,}219 \quad \text{and} \quad n \;\approx\; \frac{2}{(40 - 20\sqrt{2})\epsilon^2} \;\approx\; 1{,}707{,}106{,}781,$$

which are again remarkably close to the actual values. The Taylor series expansions are quite effective at helping us approximate the number of iterates required to be within ϵ of the attracting cycle.