MATH 374, Dynamical Systems, Fall 2017

Computer Project #1

Rates of Convergence

DUE DATE: Friday, Sept. 22, 5:00 pm

The goal of the project is for you to investigate the rate of convergence of an orbit to an attracting fixed point or period 2-cycle. In particular, we will explore the relation between the magnitude of the derivative at an attracting fixed point (or periodic cycle) and how fast (or slow) the convergence is for nearby orbits. This lab is based on Experiment 5.6 of Devaney's text.

It is **required** that you work in a group of two or three people. Any help you receive from a source other than your lab partner(s) should be acknowledged in your report. For example, a textbook, web site, another student, etc. should all be appropriately referenced at the end of your report. The project should be **typed** although you do not have to typeset your mathematical notation. For example, you can leave space for a graph, computations, tables, etc. and then write it in by hand later. You can also include graphs or computations in an appendix at the end of your report. Your presentation is important and I should be able to clearly read and understand what you are saying. Spelling mistakes and sentence fragments, for example, should not occur. Only **one project per group** need be submitted.

Your report should provide coherent answers to each of the following questions. Be sure to answer all of the questions asked. Read carefully. **Note:** Please do not overload your report (or my attention for reading) by including large numbers of graphs or any long lists of orbits.

Using Maple to Iterate Functions

Although you can use the java applet *The Function Iterator* on the Dynamical Systems and Technology Project website at Boston University (linked from our course website), Maple is far more accurate in computing orbits and much easier to use once you know the correct commands. If you would like to write some code using another language or software (e.g., C++ or Matlab), feel free to do so. Since iteration is a recursive process, computers are particularly well-suited for computing orbits.

For example, to iterate the function $f(x) = x^2 + 0.3$ with a starting value of $x_0 = 0.2$ for $n = 15$ iterations, type

```
f := x -> x^2 + 0.3:  x0 := 0.2:   n := 15:

for i from 1 to n do
     x0:= f(x0);
end do;
```

You should see the first 15 iterates of the orbit of x_0 under the function f, with the last output being `x0 := 5080.367121`

First, note that we begin by defining f, x_0, and n. These are functions and parameters that you may need to change for each problem. Be sure to execute these commands before doing the next problem, otherwise you may be iterating the wrong function or wrong initial seed.

Second, we define a `for` loop to actually compute the orbit. **Important Tip:** To skip to the next line in Maple without executing the command use the `shift` and `return` keys together. The entire `for` loop should be written as one command (at only one execution prompt `>`). If you want to learn more about `for` loops in Maple type `?for` at a command prompt.

Third, to suppress the entire output, use a colon `:` instead of a semi-colon `;` in the last two lines of the `for` loop. Typing `x0;` then gives the last value obtained from the loop. This can be useful for those orbits which converge very slowly to the fixed point, otherwise you will be scrolling through pages and pages of data.

Another useful command you may want to use is the `solve` command, used to solve equations. For example, to find the fixed points of a function f we need to solve $f(x) = x$ which is accomplished with:

```
solve(f(x)=x,x);
```

However, it is important to realize that many equations cannot be solved explicitly (e.g., with radicals). In this case you can replace `solve` with `fsolve` to find a numerical answer. For example, the real fixed points of $f(x) = x^6 + 3x + 1$ are found by typing

```
fsolve(x^6 + 3*x + 1=x,x);
```

You should obtain -1 and -0.5086603916. If you want more digits you can type `Digits := 20`, which provides 20 digits to the right of the decimal.

Lab Questions

Each of the functions listed below has an attracting fixed point or a neutral fixed point which attracts in at least one direction. The orbit of 0.2 is attracted to this fixed point (call it p). Using a computer, determine how long it takes the orbit of 0.2 to come within $\epsilon = 1 \times 10^{-5} = 0.00001$ of the fixed point. The goal is to determine how the value of the derivative at the fixed point relates to the rate of convergence for nearby orbits.

For each function, provide the following information, being as specific as possible:

a. The exact value (estimate if necessary) of the attracting or neutral fixed point p.

b. The type of fixed point (attracting, super-attracting, weakly attracting, neutral, etc.)

c. $|f'(p)|$

d. The number of iterations necessary for the orbit of 0.2 to come within ϵ of p. Specifically, find the **smallest** natural number n such that

$$|f^n(0.2) - p| \ < \ \epsilon \ = \ 1 \times 10^{-5}$$

NOTE: Please do not turn in the orbit, just the value of n. Also, for the slower converging orbits, you should set `Digits := 20` in Maple to get the best accuracy. Some of these examples take a very, very long time to converge. In these cases, try and find a good lower bound for n if you can't find the exact value.

The functions to consider are:

1. $f(x) = x^2$
2. $f(x) = x^2 + 0.09$

3. $f(x) = x^2 + 0.24$
4. $f(x) = x^2 + 0.25$
5. $f(x) = x^2 - 0.75$
6. $f(x) = 0.7x(1 - x)$
7. $f(x) = -0.7x(1 - x)$
8. $f(x) = 2x(1 - x)$
9. $f(x) = \cos x$
10. Given your results above, what is the relationship between the magnitude of the derivative at the fixed point p and the rate of convergence to p? Be as precise as possible using the examples above. Be sure to compare the attracting case versus the neutral case. Why is it that we call a fixed point with derivative 0 *super-attracting*?
11. What is the main difference in convergence to an atracting fixed point p between the cases where $f'(p) > 0$ and $f'(p) < 0$? Is there a major difference in the rate of convergence for derivatives of opposite sign but similar magnitude?

Attracting 2-Cycles

Each of the following functions has a period 2-cycle that attracts the orbit of 0.2. As before, focusing on the relationship between the derivative and the rate of convergence, provide the following information for each function, being as specific as possible:

- **a.** The **exact** values of the attracting or neutral periodic 2-cycle.
- **b.** The type of periodic cycle (attracting, super-attracting, weakly attracting, neutral, etc.)
- **c.** $|(f^2)'(p)|$ (where p is one of the values on the 2-cycle)
- **d.** The number of iterations n necessary for the orbit of 0.2 to be within $\epsilon = 1 \times 10^{-5} = 0.00001$ of the period 2-cycle. For orbits that take a long time to converge, try finding a good lower bound for n rather than finding the exact value.

The functions to consider are:

12. $f(x) = x^2 - 0.9$
13. $f(x) = x^2 - 1$
14. $f(x) = x^2 - 1.25$
15. Given your results above, what is the relationship between the magnitude of the derivative along a periodic cycle and the rate of convergence to the cycle? Be as precise as possible using the examples above. Be sure to compare the attracting versus neutral cases.