MATH 392: Seminar in Celestial Mechanics
Homework Assignment #10 (Last One!)
DUE DATE: Thurs., April 10, start of class.

Homework should be turned in at the BEGINNING OF CLASS. You should write up solutions neatly to all problems, making sure to show all your work. For this assignment you are permitted to work in groups of four or less. You only need to turn in one assignment per group. All members of the group will receive the same grade. As usual, please cite any references (web based or text) that you may have used for assistance with the assignment.

1. Read Chapters 10 and 11 of *Celestial Mechanics: The Waltz of the Planets*, by Celletti and Perozzi. What is the current definition of a "planet"? Why is it difficult to locate exoplanets (a planet outside our solar system)? What are some of the methods used to overcome these difficulties?

2. Recall from class the change to *Jacobi coordinates* in the three-body problem:

$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{q}_2 - \mathbf{q}_1 & \mathbf{v}_1 &= \tfrac{1}{m_1+m_2}\left(m_1\mathbf{p}_2 - m_2\mathbf{p}_1\right)\\
\mathbf{u}_2 &= \mathbf{q}_3 - \tfrac{1}{m_1+m_2}\left(m_1\mathbf{q}_1 + m_2\mathbf{q}_2\right) & \mathbf{v}_2 &= \tfrac{-m_3}{M}\left(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3\right) + \mathbf{p_3}\\
\mathbf{u}_3 &= \tfrac{1}{M}\left(m_1\mathbf{q}_1 + m_2\mathbf{q}_2 + m_3\mathbf{q}_3\right) & \mathbf{v}_3 &= \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3.
\end{aligned}$$

where $M = m_1 + m_2 + m_3$ is the total mass. Define

$$H(\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2) = \frac{||\mathbf{v}_1||^2}{2M_1} + \frac{||\mathbf{v}_2||^2}{2M_2} - \frac{Gm_1m_2}{||\mathbf{u}_1||} - \frac{Gm_1m_3}{||\mathbf{u}_2 + M_3\mathbf{u}_1||} - \frac{Gm_2m_3}{||\mathbf{u}_2 + M_4\mathbf{u}_1||}$$

where

$$M_1 = \frac{m_1m_2}{m_1+m_2}, \quad M_2 = \frac{m_3(m_1+m_2)}{M}, \quad M_3 = \frac{m_2}{m_1+m_2}, \quad \text{and } M_4 = \frac{-m_1}{m_1+m_2}.$$

a) Show that $\mathbf{u}_2 + M_3\mathbf{u}_1 = \mathbf{q}_3 - \mathbf{q}_1$ and $\mathbf{u}_2 + M_4\mathbf{u}_1 = \mathbf{q}_3 - \mathbf{q}_2$.

b) Show that under this change of variables, the new equations of motion are Hamiltonian, with H as the Hamiltonian function. (This problem was begun in class.)

c) Show that in these new coordinates, the moment of inertia becomes

$$I = \frac{1}{2}\left(M_1||\mathbf{u}_1||^2 + M_2||\mathbf{u}_2||^2\right).$$

Hint: Try using mutual distances as coordinates.

3. Recall that the $2m \times 2m$ matrix J is defined as

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

where I is the $m \times m$ identity matrix. A matrix A is called *symplectic* if it satisfies

$$A^T JA = J.$$

Symplectic matrices play a central role in the theory of Hamiltonian systems. Any change of variables that is symplectic (meaning that its Jacobian is a symplectic matrix) preserves the Hamiltonian structure of the original system. Denote the set of all $2m \times 2m$ symplectic matrices as $Sp(m, \mathbb{R})$.

a) Using the formula above, show that a 2×2 symplectic matrix is simply a matrix whose determinant is one.

b) Show that, in general, the determinant of a symplectic matrix is either 1 or -1. (It turns out that it is always positive one, but this is harder to prove.) Conclude that a symplectic matrix is invertible.

c) Find a formula for the inverse of a symplectic matrix involving J. Using your formula, show that the inverse of a symplectic matrix is also symplectic.

d) Show that the product of two symplectic matrices is also symplectic.

e) Using the above results, show that $Sp(m, \mathbb{R})$ is a group under matrix multiplication.

4. Consider the coordinate changes (first Jacobi, then the Hopf map) used by Chenciner and Montgomery in their proof of the existence of the figure-eight orbit in the planar, equal mass, three-body problem. The masses are each set to one and the center of mass is at the origin. Only the position components are relevant to this problem.

$$\begin{aligned}
\mathbf{u}_1 &= \tfrac{1}{\sqrt{2}}(\mathbf{q}_3 - \mathbf{q}_2) \\
\mathbf{u}_2 &= \sqrt{\tfrac{2}{3}}(\mathbf{q}_1 - \tfrac{1}{2}(\mathbf{q}_2 + \mathbf{q}_3)) \\
\mathbf{u}_3 &= \tfrac{1}{3}(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3)
\end{aligned}$$

Followed by:

$$\begin{aligned}
w_1 &= ||\mathbf{u}_1||^2 - ||\mathbf{u}_2||^2 & \qquad w_2 &= 2(\mathbf{u}_1 \cdot \mathbf{u}_2) \\
w_3 &= 2(\mathbf{u}_1 \times \mathbf{u}_2) & \qquad w_4 &= \arg(\mathbf{u}_1)
\end{aligned}$$

a) Dot the vector $\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3$ with itself to derive the identity

$$-2(\mathbf{q}_1 \cdot \mathbf{q}_2 + \mathbf{q}_1 \cdot \mathbf{q}_3 + \mathbf{q}_2 \cdot \mathbf{q}_3) = ||\mathbf{q}_1||^2 + ||\mathbf{q}_2||^2 + ||\mathbf{q}_3||^2.$$

b) Show that under the first change of variables, the moment of inertia is transformed to $2I = ||\mathbf{u}_1||^2 + ||\mathbf{u}_2||^2$. Then, show this equation is transformed to $2I = \sqrt{w_1^2 + w_2^2 + w_3^2}$ under the second change of variables. Conclude that setting $2I = 1$ gives the unit sphere. This sphere is called the *shape sphere* and is the space of similarity classes of oriented triangles. In other words, a point on the shape sphere corresponds to an equivalence class of similar triangles.

c) Show that $w_3 = 0$ if and only if the configuration of bodies is collinear. Thus, the collinear configurations all lie on the equator.

d) Show that the equilateral triangles lie at the North and South poles of the shape sphere.

e) Show that an isosceles triangle with $\mathbf{q}_1$ at the apex corresponds to the great circle $w_2 = 0$ (the isosceles meridian.) This is the shape of the figure-eight after one-twelfth of its orbit.