MATH 392: Seminar in Celestial Mechanics
Homework Assignment #2
Solution to Problem #3

Theorem: Suppose that $\mathbf{q}_0 \times \mathbf{v}_0 = \mathbf{c} \neq \mathbf{0}$ where $\mathbf{q}_0 = \mathbf{q}(0)$ and $\mathbf{v}_0 = \mathbf{v}(0)$ are the initial position and velocity, respectively. Letting $c = ||\mathbf{c}||$, there exists an orthogonal matrix A and a change of variables $\mathbf{x} = A\mathbf{q}$ such that the central force problem is converted to

$$\ddot{\mathbf{x}} = -\frac{f(r)}{r}\,\mathbf{x}$$

where $r = ||\mathbf{x}||$ and the new angular momentum is simply $(0, 0, c)$.

Proof:

We will construct the 3×3 matrix A using the vectors $\mathbf{q}_0$ and $\mathbf{c}$. First, we explain the importance of A being orthogonal. Recall that an orthogonal matrix A is one for which $A^T A = AA^T = I$. This is equivalent to having the rows and columns each forming an orthonormal basis for $\mathbb{R}^3$ (length one and mutually orthogonal). The key fact about orthogonal matrices is that they preserve lengths of vectors and angles between vectors. In other words, thinking of A as representing a linear map, the image of vectors under this map preserves lengths and angles.

To see this, write the dot product $\mathbf{v} \cdot \mathbf{w}$ between any two vectors $\mathbf{v}$ and $\mathbf{w}$ as

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{v^T w}$$

interpreting the resulting 1×1 matrix as a scalar. Then we have, for any orthogonal matrix A,

$$||A\mathbf{v}||^{\mathbf{2}} = (\mathbf{Av}) \cdot (\mathbf{Av}) = (\mathbf{Av})^{\mathbf{T}}(\mathbf{Av}) = \mathbf{v^T A^T A v} = \mathbf{v^T v} = ||\mathbf{v}||^{\mathbf{2}}$$

so that $||A\mathbf{v}|| = ||\mathbf{v}||$ and the length of $\mathbf{v}$ is unchanged under the linear map A. Furthermore, using the fact that $\mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}||\,||\mathbf{w}||\cos(\theta)$ where $\theta \in [0, \pi]$ is the angle between $\mathbf{v}$ and $\mathbf{w}$, we see that

$$\cos(\theta) = \frac{(A\mathbf{v}) \cdot (\mathbf{Aw})}{||A\mathbf{v}||\,||\mathbf{Aw}||} = \frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{v}||\,||\mathbf{w}||}.$$

This shows that the angle between $A\mathbf{v}$ and $A\mathbf{w}$ is the same as the angle between $\mathbf{v}$ and $\mathbf{w}$, although the orientation may be reversed if $\det(A) = -1$.

Using these properties, it is easy to see why the change of variables $\mathbf{x} = A\mathbf{q}$ leads to the identical Kepler problem but with $\mathbf{x}$ replacing $\mathbf{q}$. Since $A^T = A^{-1}$, we have

$$\ddot{\mathbf{x}} = A\ddot{\mathbf{q}} = A\left(-\frac{f(r)}{r}\right)\mathbf{q} = -\frac{f(r)}{r}\,AA^T\mathbf{x} = -\frac{f(r)}{r}\,\mathbf{x}$$

where $r = ||\mathbf{q}|| = ||\mathbf{A^T x}|| = ||\mathbf{x}||$ is unchanged since the orthogonal matrix A^T preserves lengths.

Since $\dot{\mathbf{x}} = A\dot{\mathbf{q}}$, the angular momentum in the new coordinates will be the vector

$$\mathbf{x}(0) \times \dot{\mathbf{x}}(0) = (A\mathbf{q}(0)) \times (A\dot{\mathbf{q}}(0)) = (A\mathbf{q}_0) \times (A\mathbf{v}_0).$$

Since A is orthogonal, the new angular momentum will have the same length c. This follows from

$$||\mathbf{x}(0) \times \dot{\mathbf{x}}(0)|| = ||(A\mathbf{q}_0) \times (A\mathbf{v}_0)|| = ||A\mathbf{q}_0||\,||A\mathbf{v}_0||\sin\theta = ||\mathbf{q}_0||\,||\mathbf{v}_0||\sin\theta = ||\mathbf{q}_0 \times \mathbf{v}_0|| = c$$

where the angle θ between $\mathbf{q}_0$ and $\mathbf{v}_0$ is unchanged under the map A. Thus if we can construct A so that the first two coordinates of $\mathbf{x}(0) \times \dot{\mathbf{x}}(0)$ are both zero, the third coordinate will either be c or $-c$ so that the length remains fixed at c.

To have the cross product of two vectors lying in the z-direction only, we want the vectors to be in the xy-plane. Thinking of how matrix multiplication works, it follows that we want the third row of A to be orthogonal to the vectors $\mathbf{q}_0$ and $\mathbf{v}_0$. But $\mathbf{c}$ is such a vector! Choose the third row of A to be the unit vector $\mathbf{c}/c$. The remaining two rows of A must be orthogonal to $\mathbf{c}$ and to themselves. A natural choice is to choose $\mathbf{q}_0$ (orthogonal to $\mathbf{c}$) and $\mathbf{c} \times \mathbf{q}_0$ (orthogonal to both $\mathbf{c}$ and $\mathbf{q}_0$ by the definition of the cross product). The reason for choosing $\mathbf{c} \times \mathbf{q_0}$ and not $\mathbf{q}_0 \times \mathbf{c}$ will be made clear in a moment. Note that $||\mathbf{c} \times \mathbf{q}_0|| = c||\mathbf{q}_0||$ since the vectors are orthogonal.

In sum, the three rows of A are given by

$$A = \begin{bmatrix} \frac{\mathbf{q}_0}{||\mathbf{q}_0||} \\ \frac{\mathbf{c}}{c} \times \frac{\mathbf{q}_0}{||\mathbf{q}_0||} \\ \frac{\mathbf{c}}{c} \end{bmatrix}$$

(each vector is really transposed so that it becomes a row rather than a column vector). By construction, A is an orthogonal matrix with $A\mathbf{q}_0$ and $A\mathbf{v}_0$ lying in the xy-plane. This implies that

$$A\mathbf{q}_0 \times A\mathbf{v}_0 = [0\;0\;\pm c]^T.$$

It remains to show that the third component of this vector is in fact just c. This can be done by showing that $\det(A) = 1$ or more directly by actually calculating the cross product.

Interpreting matrix multiplication via the dot product, we see that $A\mathbf{q}_0 = [r_0\;0\;0]^T$ where $r_0 = ||\mathbf{q}_0||$. The vector $A\mathbf{v}_0$ is

$$A\mathbf{v}_0 = \begin{bmatrix} \frac{\mathbf{q_0} \cdot \mathbf{v_0}}{r_0} \\ \frac{c}{r_0} \\ 0 \end{bmatrix}$$

where the second component follows using the vector identity from question #2:

$$\left(\frac{\mathbf{c}}{c} \times \frac{\mathbf{q}_0}{r_0}\right) \cdot \mathbf{v}_0 = \frac{\mathbf{c}}{c} \cdot \left(\frac{\mathbf{q}_0}{r_0} \times \mathbf{v}_0\right) = \frac{\mathbf{c}}{c} \cdot \frac{\mathbf{c}}{r_0} = \frac{c}{r_0}.$$

Thus, computing the cross product of these two vectors gives

$$A\mathbf{q}_0 \times A\mathbf{v}_0 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ r_0 & 0 & 0 \\ \frac{\mathbf{q_0} \cdot \mathbf{v_0}}{r_0} & \frac{c}{r_0} & 0 \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix}$$

as desired. □