MATH 392: Seminar in Celestial Mechanics
Homework Assignment #6

DUE DATE: Thurs., Feb. 28, start of class.

Homework should be turned in at the BEGINNING OF CLASS. You should write up solutions neatly to all problems, making sure to show all your work. You are strongly encouraged to work on these problems with other classmates, although the solutions you turn in should be your **own** work. Please cite any references (web based or text) that you may have used for assistance with the assignment.

Note: Please list the names of any students or faculty who you worked with on the top of the assignment.

1. Read Chapter 6 of *Celestial Mechanics: The Waltz of the Planets*, by Celletti and Perozzi.

2. **Euler's Theorem for Homogeneous Potentials:** A vector-valued function $U(\mathbf{q})$, $U : \mathbb{R}^m \mapsto \mathbb{R}$ satisfying
$$U(\lambda \mathbf{q}) \;=\; \lambda^k U(\mathbf{q}) \tag{1}$$
for all positive scalars λ and some fixed real constant k is called a **homogeneous function of order** k.

   a) Show that the Newtonian potential function U of the n-body problem is homogeneous of order -1.

   b) Prove Euler's Theorem for Homogeneous Potentials: If $U(\mathbf{q})$ is homogeneous of order k, then
$$\nabla U(\mathbf{q}) \cdot \mathbf{q} \;=\; kU(\mathbf{q}).$$
   The left-hand side is the dot product of two vectors. *Hint:* Carefully differentiate both sides of equation (1) with respect to λ and evaluate at $\lambda = 1$.

   c) Conclude that the Newtonian potential function U of the n-body problem satisfies
$$\nabla U(\mathbf{q}) \cdot \mathbf{q} \;=\; -U(\mathbf{q}).$$
   Use this to prove that there are no equilibrium points in the n-body problem (Ex. 2.1 in Pollard).

3. **The Ideal Pendulum:** This problem explores the case of an ideal pendulum (no friction) moving under the influence of gravity only. This is a reasonable model for a well-built and well-lubricated pendulum. We will assume that the ratio between the gravitational constant and the length of the pendulum's arm is one. The only position variable is θ representing the angle the pendulum makes with the downward vertical axis. If $\theta > 0$, the pendulum is to the right of downward vertical. Since θ is an angular variable, we really have $-\pi \leq \theta \leq \pi$ and thus the phase space is a cylinder $\{(\theta, v) : \theta \in [-\pi, \pi], v \in \mathbb{R}\}$. However, since its a bit challenging to draw on a cylinder, use the θv-plane as the phase space and think of identifying solutions crossing the vertical line $\theta = \pi$ with those crossing $\theta = -\pi$.

The equation of motion is simply $\ddot{\theta} + \sin\theta = 0$, or as a 1st-order system,

$$\begin{aligned} \dot{\theta} &= v \\ \dot{v} &= -\sin\theta. \end{aligned}$$

a) Find all equilibrium points of the system and describe what they represent in physical terms of the pendulum.

b) Linearize the system about each equilibrium point and classify the type of equilibrium point (ie. sink, source, saddle, etc.) Explain why your findings make sense physically in terms of stability.

c) Show that the system is Hamiltonian by finding a Hamiltonian function $H(\theta, v)$.

d) Sketch the phase plane diagram for the ideal pendulum. You may use technology for assistance but try it first without a computer.

e) There are essentially three types of solutions other than equilibrium points. One of these is very special, serving as a border or **separatrix** between the other two types. The separatrix solution connects two equilibrium points. Describe physically the behavior of the pendulum for each of the three cases. What are the values of the energy H for each case?

f) Show that it takes infinitely long to reach either equilibrium point along the separatrix solution.

g) (Hard) Find a formula for the period of the solutions near the origin and show that this period increases as you move outwards from the origin.

4. Do the following exercises from Pollard's text: **Chapter 2: 2.2, 2.3, 2.4, 3.1**.

   Some hints: In problem #2.2, use the fact that if $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)\, dx$. In problem #2.3, note that T_1 is identical to T or, in our notation, the Kinetic energy K, since the vectors $\mathbf{p}_k$ are really momentum vectors $m_k \mathbf{v}_k$.