MATH 392: Seminar in Celestial Mechanics

Homework Assignment #7

DUE DATE: Mon., March 17, start of class.

Homework should be turned in at the BEGINNING OF CLASS. You should write up solutions neatly to all problems, making sure to show all your work. You are strongly encouraged to work on these problems with other classmates, although the solutions you turn in should be your **own** work. Please cite any references (web based or text) that you may have used for assistance with the assignment.

Note: Please list the names of any students or faculty who you worked with on the top of the assignment.

1. Read Chapter 7 "Of Moon and man" of *Celestial Mechanics: The Waltz of the Planets*, by Celletti and Perozzi.

   a) What is the *synodic period* and how long is it? Why is it that if we look at the moon on our 19th, 38th, 57th, ... birthdays, it will be in the same phase as when we were born?

   b) What is the *Saros* and why is it important?

   c) What are the key mathematical ideas at work in this chapter?

2. For the n-body problem, recall that at the time $t = T$ of a total collapse, the moment of inertia $I(t)$ approaches 0 while the second derivative of $I(t)$ approaches ∞. Give an explicit example (formula) for a positive function $I(t)$ with these two properties.

3. Recall that the moment of inertia (measuring the "size" of the system) with respect to the center of mass is given by

$$I(\mathbf{q}) = \frac{1}{2} \sum_{k=1}^{n} m_k ||\mathbf{q_k}||^2.$$

   a) Assuming the center of mass is at the origin, show that I can be written in terms of the mutual distances r_{jk} as

$$I = \frac{1}{2M} \sum_{j<k}^{n} m_j m_k r_{jk}^2 \qquad (1)$$

   where $M = m_1 + m_2 + \cdots + m_n$ is the total mass. *Hint:* Start by simplifying the double sum

$$\sum_{j=1}^{n} \sum_{k=1}^{n} m_j m_k r_{jk}^2$$

   where $r_{jk} = 0$ if $j = k$.

   b) Knowing the three sides of a triangle is enough to describe the triangle up to rotation and translation (the SSS Postulate from Euclidean Geometry). Thus, the mutual distances r_{12}, r_{13} and r_{23} make ideal variables for the three-body problem, provided the configuration is non-collinear. In particular, both I and U can be expressed in terms of these variables. Using formula (1) for I, show that the **only** non-collinear central configuration in the three-body problem is the equilateral triangle. What is the length of the side of the triangle?

4. In our calculations deriving the equations for a relative equilibrium, we used the matrices $M = \text{diag}\{m_1, m_1, \ldots, m_n, m_n\}$ and $K = \text{diag}\{J, J, \ldots, J\}$ where

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

   Show that M and K commute, that is $MK = KM$, and that $K^2 = -I$.

5. Recall that for a relative equilibrium, the moment of inertia I is constant for all time.

   a) Show that both the potential function U and the kinetic energy K are constant for all time as well. How are they related to each other?

   b) Calculate the angular momentum $\mathbf{c}$ and the kinetic energy K of a relative equilibrium in terms of I. (Assume the relative equilibrium lies in the xy-plane.) Show that Sundman's inequality $c^2 \leq 4I(\ddot{I} - H)$ becomes an **equality** at a relative equilibrium.

6. Find all collinear central configurations with three equal masses. How does your answer depend on the common value of the masses m?

7. Do the following exercises from Pollard's text: **Chapter 2: 1.1**.

   Some hints: For problem #1.1, start by guessing a solution of the form $\mathbf{q_i}(t) = f(t)\mathbf{x_i}$ where $f(t)$ is some unknown scalar function and $\mathbf{x_i}$ is a vertex of an equilateral triangle. Derive two equations — one for the vectors $\mathbf{x_i}$ (algebraic) and the other for the function $f(t)$ (differential). You should recognize both equations!