MATH 392: Seminar in Celestial Mechanics
Homework Assignment #9
DUE DATE: Thurs., April 3, start of class.

Homework should be turned in at the BEGINNING OF CLASS. You should write up solutions neatly to all problems, making sure to show all your work. For this assignment you are permitted to work in groups of three or less. You only need to turn in one assignment per group. All members of the group will receive the same grade. As usual, please cite any references (web based or text) that you may have used for assistance with the assignment.

1. Read Chapter 9 "Lords of the rings" of *Celestial Mechanics: The Waltz of the Planets*, by Celletti and Perozzi.

   a) What was the Astronomer Royal, George Biddell Airy referring to when he exclaimed "It is one of the most remarkable applications of mathematics to physics that I have ever seen."?

   b) What is the Roche limit R_L and why is it significant to the formation of planetary rings?

   c) Compute the Roche limit (in km) for Mars assuming the orbiting body is a comet with density $0.5g/cm^3$.

   d) Why is it problematic if the density of the planet is less than half the density of the orbiting body?

2. Recall that the $2m \times 2m$ matrix J is defined as

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

   where I is the $m \times m$ identity matrix. Show that $\det(J) = 1$ and $J^2 = -I$.

3. Recall that a *Hamiltonian matrix* A is a $2m \times 2m$ matrix of the form $A = JS$ where J is given above and S is a symmetric matrix of the appropriate size. Suppose that A and B are Hamiltonian matrices. Denote $sp(m, \mathbb{R})$ as the set of all real $2m \times 2m$ Hamiltonian matrices.

   a) Show that αA, A^T and $A + B$ are also Hamiltonian matrices for any scalar α. This shows that $sp(m, \mathbb{R})$ is a vector space (the "vectors" are Hamiltonian matrices.)

   b) Show that the trace of a Hamiltonian matrix is always zero.

   c) By way of an example, show that $sp(m, \mathbb{R})$ is **not** closed under matrix multiplication.

   d) Show that $[A, B] = AB - BA$ (the *Lie Product*) is also a Hamiltonian matrix. This implies that $sp(m, \mathbb{R})$ is a *Lie Algebra*.

   e) Find the dimension of $sp(m, \mathbb{R})$ in terms of m. In other words find the dimension of the vector space or simply the number of "free variables" in a generic $2m \times 2m$ Hamiltonian matrix.

4. Suppose that $\mathbf{x}_0$ is an equilibrium point of a Hamiltonian system and the Jacobian evaluated at $\mathbf{x}_0$ is the Hamiltonian matrix A. If $\det(A) < 0$, show that the equilibrium point is unstable. *Hint:* Recall that the product of the eigenvalues of a matrix is equal to the determinant of the matrix.