Composing with Numbers: Sir Peter Maxwell Davies and Magic Squares

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Sir Peter Maxwell Davies

- Born in Salford, Lancashire England, 1934.
- Studied at University of Manchester and Royal Manchester College of Music. Helped form the group *New Music Manchester*, committed to contemporary music.
- Studied in Rome, Princeton, and Australia before settling on the Orkney Islands off the coast of Scotland.
- Served as the associate conductor/composer with the BBC Philharmonic Orchestra from 1992-2002.
- Has conducted major orchestras such as the Cleveland and Boston Symphony Orchestras.

Sir Peter Maxwell Davies (cont.)

- Knighted in 1987 (thus the Sir honorific).
- In 2004 appointed Master of the Queen's Music.
- Awarded Honorary Doctorate of Music by Oxford University in 2005.
- One of the first classical composers to open a music download website `maxopus.com` (1996) promoting his works.
- Known for using magic squares as an architectural blueprint for creating structures in his music, even generating the music itself.
- Example: *Ave Maris Stella* (1975): 9×9 magic square associated with the moon to permute notes of a plainchant melody and to determine durations of the notes.

Definition

A square array of numbers where each row, column, and main diagonal sums up to the same amount is called a magic square. An $n \times n$ magic square uses only the numbers $1, 2, 3, 4, \cdots, n^2 - 2, n^2 - 1, n^2$.

An $n \times n$ matrix is a square array of numbers with n rows and n columns.

Example: $(n = 3)$

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

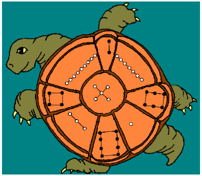
This is <u>not</u> a magic square.

The case 3×3

8	1	6
3	5	7
4	9	2

- Each row sums to 15.
- Each column sums to 15.
- Each of the two main diagonals sums to 15.
- MAGIC!!!
- The number 15 is called the magic constant (particular to $n = 3$.)
- Up to rotation and reflection, (think D_4), this is the only possible 3×3 magic square.
- The square was known to the ancient Chinese (over 3,000 years ago), who called it Lo Shu.

Legend of the Lo Shu Magic Square



- While walking along the bank of the Lo River (or Yellow River), Emperor Yu watched a mystical turtle crawl out of the water.
- The turtle seemed normal except for a series of dots in each panel of its shell. Remarkably, Yu noticed that the total number of dots in any row, column, or diagonal was always the same!
- The shell pattern became known as Lo Shu (the word “shu” means book or scroll).
- The emperor kept the turtle as a mathematical pet, where it became world renown, earning visits from mathematicians and kings alike. Feng Shui formulas of astrology and I-Ching are based on the Lo Shu magic square.

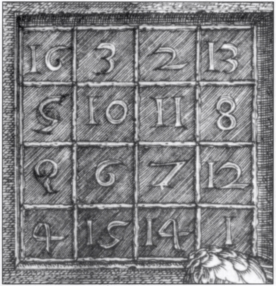


Figure : Dürer's Melancholia I, 1514

Dürer's 4×4 Magic Square

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

- The sum of any row, column, or main diagonal is 34. Magic!!
- The sum of the four corners or the four middle squares is also 34.
- 1514 engraving Melancholia I by German artist (painter, printmaker, engraver) and mathematician Albrecht Dürer featured the above magic square in the upper-right corner.
- The date of the engraving appears in the middle of the bottom row of the magic square.
- Up to symmetry, there are a total of 880 "different" 4×4 magic squares.

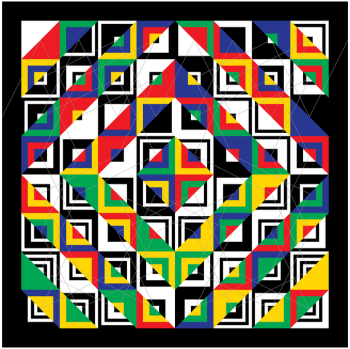


Figure : The pseduo-magic square on the front of Gaudí's Sagrada Família (left; rows, columns, and diagonals sum to 33), and Magic Square 8 Study: A Breeze over Gwalior (right), a work based on an 8 $\times$ 8 magic square by mathematically-inspired artist Margaret Kepner.

What about a 2×2 magic square?

Try $n = 2$.

1	2
3	4

1	2
4	3

1	3
2	4

1	3
4	2

1	4
2	3

1	4
3	2

Conclusion: There is no 2×2 magic square.

The Magic Constant

Nice Fact: The rows, columns, and main diagonal of any $n \times n$ magic square sum up to the same number M_n, called the magic constant. This number changes (increases) with n and is given by

$$M_n = \frac{n(n^2+1)}{2}.$$

n	M_n
2	5
3	15
4	34
5	65
6	111
7	175
8	260
9	369
10	505

A Mirror of Whitening Light

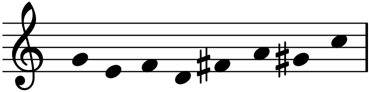
- Composed by Davies in 1977.
- Latin title: “Speculum Luminis de Albensis”.
- Commissioned by the London Sinfonietta (instrumental ensemble).
- Dedicated to the American composer Roger Sessions (one of his teachers).
- Title refers to alchemy, in particular the process of “whitening” a base metal into gold.
- Inspired by the remarkable properties of light, as noticed while working in Orkney.
- Uses the 8×8 magic square of Mercury to create entire set of notes and durations for the piece.

The Architectural Blueprint

8	58	59	5	4	62	63	1
49	15	14	52	53	11	10	56
41	23	22	44	45	19	18	48
32	34	35	29	28	38	39	25
40	26	27	37	36	30	31	33
17	47	46	20	21	43	42	24
9	55	54	12	13	51	50	16
64	2	3	61	60	6	7	57

Table : The 8×8 magic square of Mercury.

Davies takes 8 notes from the plainchant Veni Sancte Spiritus (Come Holy Spirit), often called the "Golden Sequence," to create a melodic phrase with 8 distinct notes. He then transposes the phrase to start on each of the 8 notes, yielding an 8×8 matrix of notes. This matrix is then mapped onto the magic square of Mercury. Different paths through the resulting magical music square generate the pitches and rhythmic durations for the piece.



G 1	E 2	F 3	D 4	$F^\sharp$ 5	A 6	$G^\sharp$ 7	C 8
E 9	$C^\sharp$ 10	D 11	B 12	$D^\sharp$ 13	$F^\sharp$ 14	F 15	A 16
F 17	D 18	$E^\flat$ 19	C 20	E 21	G 22	$G^\flat$ 23	$B^\flat$ 24
D 25	B 26	C 27	A 28	$C^\sharp$ 29	E 30	$E^\flat$ 31	G 32
$F^\sharp$ 33	$D^\sharp$ 34	E 35	$C^\sharp$ 36	F 37	$G^\sharp$ 38	G 39	B 40
A 41	$F^\sharp$ 42	G 43	E 44	$A^\flat$ 45	B 46	$B^\flat$ 47	D 48
$G^\sharp$ 49	F 50	$F^\sharp$ 51	$D^\sharp$ 52	G 53	$A^\sharp$ 54	A 55	$C^\sharp$ 56
C 57	A 58	$A^\sharp$ 59	G 60	B 61	D 62	$C^\sharp$ 63	F 64

Figure : The eight-note phrase (above) used by Davies in A Mirror of Whitening Light, derived from the plainchant Veni Sancte Spiritus. The notes are transposed repeatedly to fill an 8×8 matrix, called a transposition square.

C	A	A$^\sharp$	F$^\sharp$	D	D	C$^\sharp$	G
G$^\sharp$	F	F$^\sharp$	D$^\sharp$	G	D	C$^\sharp$	C$^\sharp$
A	G$^\flat$	G	E	A$^\flat$(D)	E$^\flat$	D	D (A$^\flat$)
G	D$^\sharp$	E	C$^\sharp$	A (D)	G$^\sharp$	G	D (A)
B	B	C	F	C$^\sharp$	E	E$^\flat$	F$^\sharp$
F	B$^\flat$	B	C	E	G	F$^\sharp$	B$^\flat$
E	A	A$^\sharp$	B	D$^\sharp$	F$^\sharp$	F	A
F	E	F	B	G	A	G$^\sharp$	C

Table : Davies' mapping of the plainchant notes onto the 8×8 magic square of Mercury. The notes in parentheses are adjustments to the square that were used by the composer.

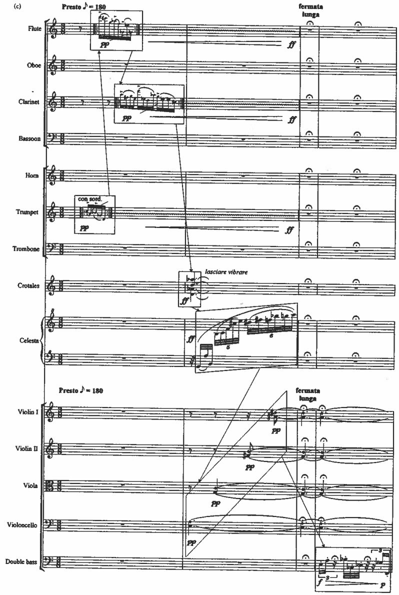


Figure 5. Peter Maxwell Davies, *A mirror of whitening light.*

Figure : Opening to A Mirror of Whitening Light

Opening Notes

Trumpet: C A B♭ G♭ D

Flute: D C♯ G A♭ F G♭ E♭ G

These are, in order, the first 13 pitches of the top two rows.

Clarinet: A F♯ G E (D) E♭ D (A♭) G E♭ E C♯

Crotales: (antique cymbals) (D) A♭

Celesta: G (A) B B C F C♯ E D♯ F♯ F B♭ B C

These are (with a few exceptions shown in parentheses), in order, the pitches for the next three and a half rows of the magic square.

Figure : The flute part starting at rehearsal letter Z in A Mirror of Whitening Light. The notes are obtained from the magic square by spiraling around the square in the counterclockwise direction.

Rhythms Determined by the Magic Square modulo 8

- Consider the durations of each note in the previous flute part. Count one for each eighth note: a quarter note counts as 2, a dotted quarter as 3, a half note as 4, etc.
- Important: Include the length of a rest with the preceding note. Thus, an eighth note followed by a quarter note rest should count as 3.
- The durations of the notes of the flute part form the sequence 7, 6, 4, 5, 3, 2, 8. These are precisely the numbers in the top row of the magic square reduced modulo 8, with 0 replaced by an 8. The rhythmic durations for the flute part follow the same spiral pattern through the magic square as the notes!
- Analyzing the patterns in A Mirror of Whitening Light is a bit like doing a musical “word-search.”

8	2	3	5	4	6	7	1
1	7	6	4	5	3	2	8
1	7	6	4	5 (8)	3	2	8 (5)
8	2	3	5	4 (1)	6	7	1 (4)
8	2	3	5	4	6	7	1
1	7	6	4	5	3	2	8
1	7	6	4	5	3	2	8
8	2	3	5	4	6	7	1

Table : The 8×8 magic square of Mercury reduced modulo 8, except that any number equivalent to 0, that is, any multiple of 8, has been written as an 8. The rhythmic durations for different instruments are determined by following different paths through the square.

Davies on A Mirror of Whitening Light

"And if you go across that square of the numbers arranged in a particular way, they make very interesting patterns. And I see these patterns, in the first place, possibly as dance patterns; and one gets to know them by heart. One doesn't in fact deal with numbers at all. One deals rather as somebody who is dealing with bell-changes, with actual patterns with changes" (emphasis added).

Rigid structure imposed by the magic square?

"I firmly believe that the more one controls the flow of one's wildest inspiration, the wilder it sounds. And so when I really wanted to be wild towards the climax of this work, I imposed very rigid rhythmic and tonal controls derived from the plainsong, and from that magic square; and the result is really quite extraordinary I find, even now."