The Fibonacci Numbers and the Golden Ratio

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The Fibonacci Numbers

Definition

The Fibonacci Numbers are the numbers in the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots \; .$$

This is a recursive sequence defined by the equations

$$F_1 = 1, \; F_2 = 1, \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \quad \text{for all } n \geq 3.$$

Here, F_n represents the nth Fibonacci number (n is called an index).

Example: $F_4 = 3, \quad F_6 = 8, \quad F_{10} = 55, \quad F_{102} = F_{101} + F_{100}.$

Often called the "Fibonacci Series" or "Fibonacci Sequence".

The Fibonacci Numbers: History

- Numbers named after Fibonacci by Edouard Lucas, a 19th century French mathematician who studied and generalized them.
- Fibonacci was a pseudonym for Leonardo Pisano (1175-1250). The phrase "filius Bonacci" translates to "son of Bonacci."
- Father was a diplomat, so he traveled extensively.
- Fascinated with computational systems. Writes important texts reviving ancient mathematical skills. Described later as the "solitary flame of mathematical genius during the middle ages." (V. Hoggatt)
- Imported the Hindu-arabic decimal system to Europe in his book *Liber Abbaci* (1202). Latin translation: "book on computation."

The Fibonacci Numbers: More History

- Before Fibonacci, Indian scholars such as Gopala (before 1135) and Hemachandra (1089–1172) discussed the sequence $1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$ in their analysis of Indian rhythmic patterns.

- Fibonacci Fun Fact: The number of ways to divide n beats into "long" (L, 2 beats) and "short" (S, 1 beat) pulses is F_{n+1} (see Section 1.4 of course text).

- Example: $n = 3$ has SSS, SL, or LS as the only possibilities. $F_4 = 3$.

- Example: $n = 4$ has SSSS, SLS, LSS, SSL, LL as the only possibilities. $F_5 = 5$.

- Recursive pattern is clear: To find the number of ways to subdivide n beats, take all the possibilities for $n - 2$ beats and append an L, and take those for $n - 1$ and append an S.

The Fibonacci Numbers: Popular Culture

- 13, 3, 2, 21, 1, 1, 8, 5 is part of a code left as a clue by murdered museum curator Jacque Saunière in Dan Brown's best-seller *The Da Vinci Code*.

- Crime-fighting FBI math genius Charlie Eppes mentions how the Fibonacci numbers occur in the structure of crystals and in spiral galaxies in the Season 1 episode "Sabotage" (2005) of the television crime drama *NUMB3RS*.

- The rap group *Black Star* uses the following lyrics in the song “Astronomy (8th Light)”

  Now everybody hop on the one, the sounds of the two
  It's the third eye vision, five side dimension
  The 8th Light, is gonna shine bright tonight

Fibonacci Numbers in the Comics

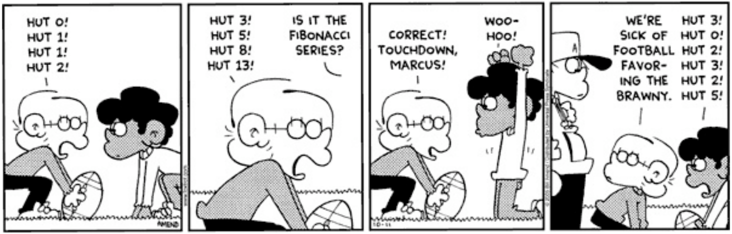
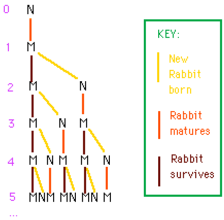


Figure : *FoxTrot* by Bill Amend (2005)

Key Passage from the 3rd section of Fibonacci's *Liber Abbaci*:

> *"A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?"*

Answer: $233 = F_{13}$. The Fibonacci numbers are generated as a result of solving this problem!



Bee Populations

- A bee colony typically has 1 female (the Queen Q) and lots of males (Drones D).
- Drones are born from unfertilized eggs, so D has one parent, Q.
- Queens are born from fertilized eggs, so Q has two parents, D and Q.

	Parents	Gr-parents	Gt-Gr-parents	Gt-Gt-Gr-p's	G-G-G-G-p's
D	1	2	3	5	8
Q	2	3	5	8	13

Table : Number of parents, grand-parents, great-grand parents, etc. for a drone and queen bee.

Fibonacci Numbers in Nature

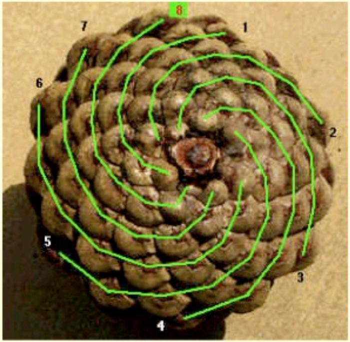
- Number of petals in “most” flowers: e.g., 3-leaf clover, buttercups (5), black-eyed susan (13), chicory (21).
- Number of spirals in bracts of a pine cone or pineapple, in both directions, are typically consecutive Fibonacci numbers.
- Number of spirals in the seed heads on daisy and sunflower plants.
- Number of leaves in one full turn around the stem of some plants.
- This is not a coincidence! Some of the facts about spirals can be explained using continued fractions and the golden ratio.

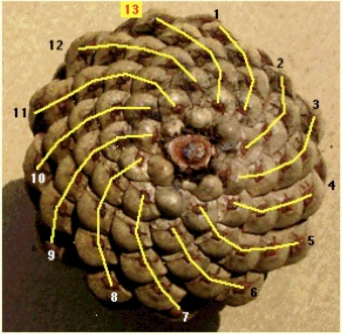

Figure : Columbine (left, 5 petals); Black-eyed Susan (right, 13 petals)


Figure : Shasta Daisy (left, 21 petals); Field Daisies (right, 34 petals)

Bracts arranged in Fibonacci numbers of spirals

Adjacent Fibonacci numbers, 8, 13





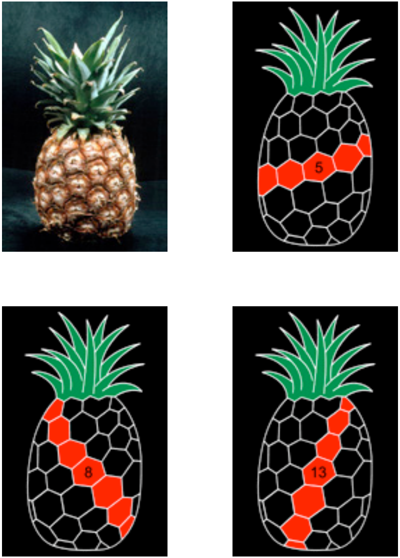
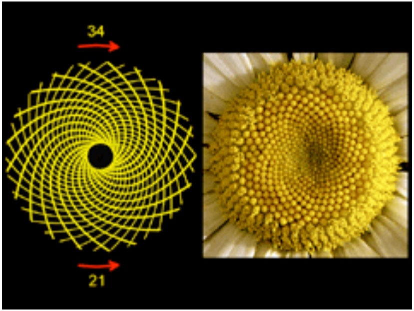


Figure : Pineapple scales often have three sets of spirals with 5, 8 and 13.



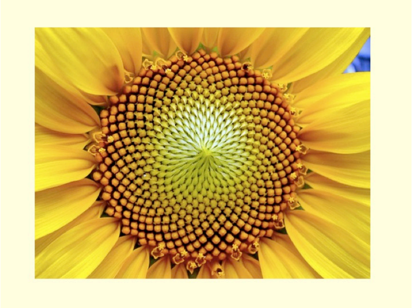
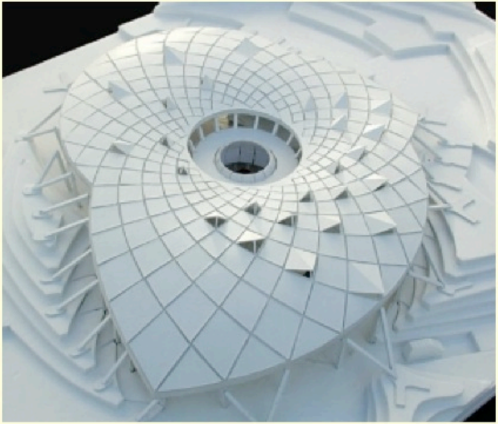


Figure : In most daisy or sunflower blossoms, the number of seeds in spirals of opposite direction are consecutive Fibonacci numbers.



Model of the Education Building, The Eden Project, Cornwall
by Joylon Brewis and Peter Randall-Page

Figure : The chimney of Turku Energia in Turku, Finland, featuring the Fibonacci sequence in 2m high neon lights (Mario Merz, 1994).

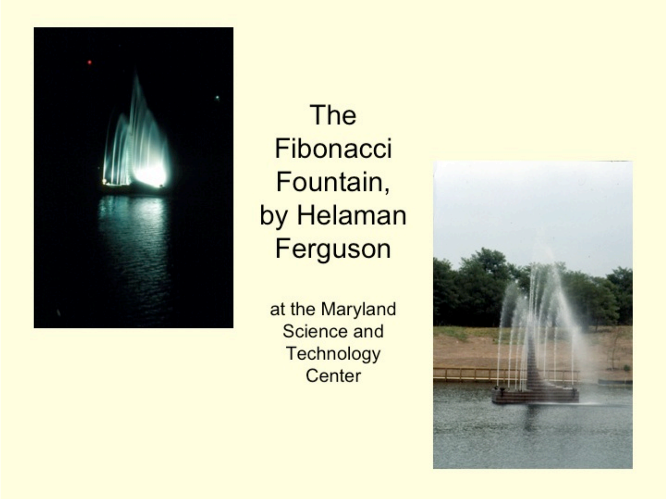


Figure : Structure based on a formula connecting the Fibonacci numbers and the golden ratio. The fountain consists of 14 (?) water cannons located along the length of the fountain at intervals proportional to the Fibonacci numbers. It rests in Lake Fibonacci (reservoir).

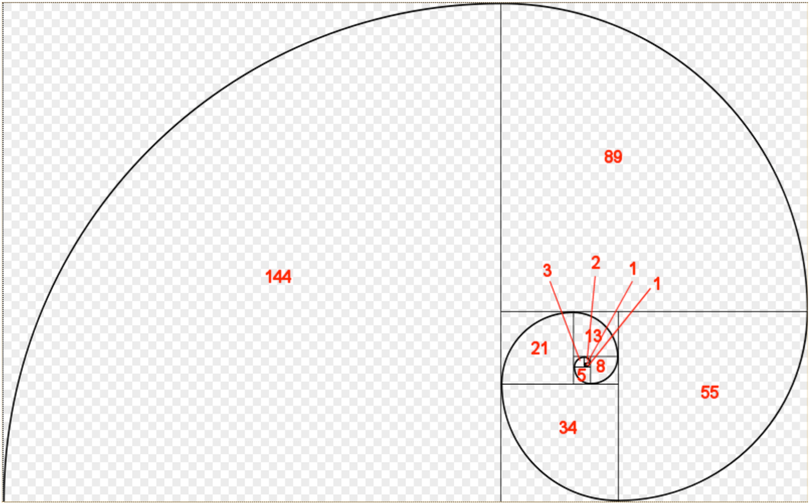


Figure : The Fibonacci Spiral, which approximates the Golden Spiral, created in a similar fashion but with squares whose side lengths vary by the golden ratio ϕ. Each are examples of Logarithmic Spirals, very common in nature.

Chambered nautilus shell

Figure : The Pinwheel Galaxy (also known as Messier 101 or NGC 5457).

Fibonacci wall art, by Dan Freund

Connections with the Golden Ratio

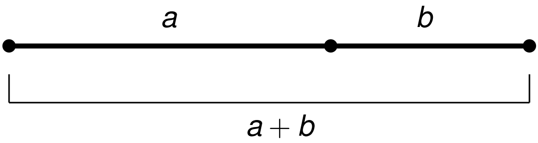


Figure : The ratio $a : b$ equals the ratio $a + b : a$, called the golden ratio.

$$\frac{a+b}{a} = \frac{a}{b} \implies \phi = \frac{a}{b} = \frac{1+\sqrt{5}}{2} \approx 1.61803398875$$

Fibonacci Fun Fact: (prove on HW #2)

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi.$$

Note: This limit statement is true for any recursive sequence with $F_n = F_{n-1} + F_{n-2}$, not just the Fibonacci sequence.

The Golden Ratio

- Other names: Golden Mean, Golden Section, Divine Proportion, Extreme and Mean Ratio
- Appears in Euclid's *Elements*, Book IV, Definition 3:

  > *A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the less.*

- Known to ancient Greeks — possibly used in ratios in their architecture/sculpture (controversial).
- Named ϕ in the mid-20th century in honor of the ancient Greek architect Phidias.

The Pentagram

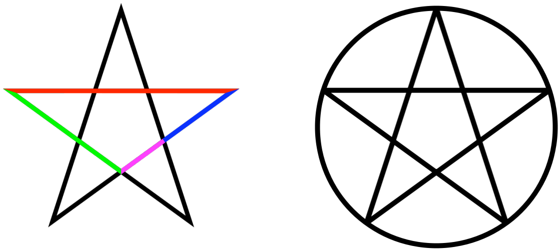
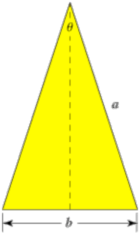


Figure : Left: the Pentagram — each colored line segment is in golden ratio to the next smaller colored line segment. Right: the Pentacle (a pentagram inscribed inside a circle.)

The pentagram is a five-pointed star that can be inscribed in a circle with equally spaced vertices (regular pentagon).

The Pythagoreans used the pentagram (called it *Hugieia*, “health”) as their symbol in part due to the prevalence of the golden ratio in the line segments.

The Golden Triangle



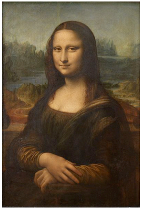
Note: The isosceles triangle formed at each vertex of a pentagram is a golden triangle. This is an isosceles triangle where the ratio of the hypotenuse *a* to the base *b* is equal to the golden ratio $\phi = a/b$.

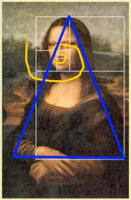
Using some standard trig. identities, one can show that

$$\theta \;=\; 2\sin^{-1}\left(\frac{b}{2a}\right) \;=\; 2\sin^{-1}\left(\frac{1}{1+\sqrt{5}}\right) \;=\; \frac{\pi}{5} \;=\; 36^\circ.$$

The two base angles are then each $2\pi/5 = 72^\circ$.

The Divine Proportion





In his book, *Math and the Mona Lisa: The Art and Science of Leonardo DaVinci*, Bulent Atalay claims that the golden triangle can be found in Leonardo da Vinci's *Mona Lisa*. Really?!

The ratio of the height to the width of the entire work is the golden ratio!

Renaissance writers called the golden ratio the divine proportion (thought to be the most aesthetically pleasing proportion).
Luca Pacioli's *De Divina Proportione* (1509) was illustrated by Leonardo da Vinci (Pacioli was his math teacher), demonstrating ϕ in various manners (e.g., architecture, perspective, skeletonic solids).

Fibonacci Phyllotaxis

In 1994, Roger Jean conducted a survey of botany literature encompassing 650 species and 12,500 specimens. He estimated that among plants displaying spiral or multijugate phyllotaxis (“leaf arrangement”) about 92% of them have Fibonacci phyllotaxis.

Question: How come so many plants and flowers have Fibonacci numbers?

Succint Answer: Nature tries to optimize the number of seeds in the head of a flower. Starting at the center, each successive seed occurs at a particular angle to the previous, on a circle slightly larger in radius than the previous one. This angle needs to be an irrational multiple of 2π, otherwise there is wasted space. But it also needs to be poorly approximated by rationals, otherwise there is still wasted space.

Fibonacci Phyllotaxis (cont.)

Figure : Seed growth based on different angles α of dispersion. Left: $\alpha = 90°$. Center $\alpha = 137.6°$. Right: $\alpha = 137.5°$.

What is so special about 137.5°? It's the golden angle!

Dividing the circumference of a circle using the golden ratio gives an angle of

$$\alpha \;=\; \pi(3 - \sqrt{5}) \;\approx\; 137.5077641°.$$

This seems to be the best angle available.

Example: The Golden Angle

Figure : The Aonium with 3 CW spirals and 2 CCW spirals. Below: The angle between leaves 2 and 3 and between leaves 5 and 6 is very close to 137.5°.

Why ϕ?

The least "rational-like" irrational number is ϕ! This has to do with the fact that the continued fraction expansion of ϕ is $[1; 1, 1, 1, 1, 1, 1, \ldots]$. (See Section 4.5.2 of the course text for an introduction to continued fractions.)

On the other hand, the convergents (the best rational approximations to ϕ) are precisely the ratios of consecutive Fibonacci numbers.

Thus, the number of spirals we see are often consecutive Fibonacci numbers. Since the petals of flowers are formed at the extremities of the seed spirals, we also see Fibonacci numbers in the number of flower petals too!

Wow! Mother Nature Knows Math.