# Change (Bell) Ringing

Gareth E. Roberts

Department of Mathematics and Computer Science
College of the Holy Cross
Worcester, MA

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*Change ringing is a non-competitive and non-violent team activity that is highly stimulating intellectually and mildly demanding physically, and makes a beautiful sound. It develops mental and physical skills in a context of communal effort. The intense concentration required brings euphoric detachment that cleanses the mind of the day's petty demands and frustrations.*

North American Guild of Change Ringers

Figure : The Swan Bells Tower in Perth, Australia, a unique icon for Western Australia. Contains 12 royal bells from England (St. Martin-in-the-Fields).

Figure : Bell ringing demonstration in Swan Bell Tower.

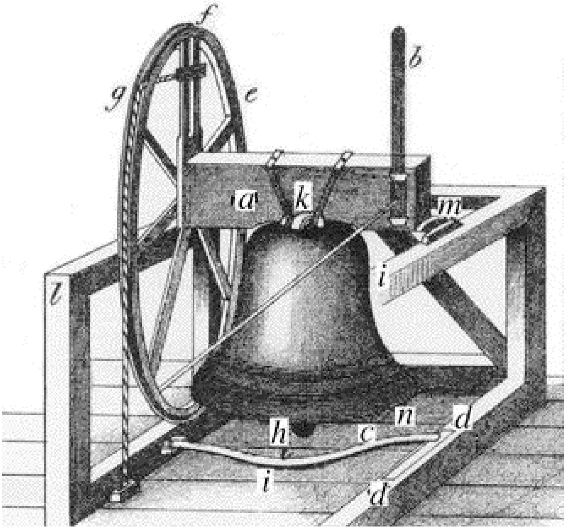



Figure : A large bell and the parts required to make it ring.

a. Stock
b. Stay
c. Slider
d. Blocks
e. Wheel
f. Groove of Wheel
g. Fillet
h. Ball of Clapper
i. Flight of Clapper
k. Cannons
l. Timber of Cage
m. Gudgeons
n. Lip of Bell


Figure : The floating belfry on the Thames river, London, during the Queen's Diamond Jubilee in June, 2012.

# Change Ringing: An Example

| | | |
|---|---|---|
| 1 2 3 4 | 1 3 4 2 | 1 4 2 3 |
| 2 1 4 3 | 3 1 2 4 | 4 1 3 2 |
| 2 4 1 3 | 3 2 1 4 | 4 3 1 2 |
| 2 4 3 1 | 3 2 4 1 | 4 3 2 1 |
| 4 2 3 1 | 2 3 4 1 | 3 4 2 1 |
| 4 2 1 3 | 2 3 1 4 | 3 4 1 2 |
| 4 1 2 3 | 2 1 3 4 | 3 1 4 2 |
| 1 4 3 2 | 1 2 4 3 | 1 3 2 4 |
| | | 1 2 3 4 |

*Canterbury Minimus* (a piece on 4 bells)

There are $4! = 24$ different possible rows. Each must be rung exactly once starting and ending with rounds (1 2 3 4).

Rules to ring an extent on $n$ bells:

1. The first and last changes (rows) are rounds ($1\ 2\ 3\ 4 \cdots n$).
2. Other than rounds, all of the other $n!$ changes occur exactly once.
3. Between successive changes, no bell moves more than one position (no jumping).
4. No bell rests for more than 2 (sometimes relaxed further to 4) positions.
5. Each working bell should do the same amount of "work" (obey the same overall pattern).
6. Horizontal symmetry should be present in the extent to help the ringers learn the path of their respective bell. This is called the palindrome property.

**Note:** Rules 1 - 3 are mandatory for an extent while Rules 4 - 6 are optional though often satisfied and desired.

- A reordering of the numbers 1 2 3 4 $\cdots n$ is called a permutation.
- How many possible changes on $n$ bells?
  **Answer:** $n!$

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$$

- The permissible "moves" between changes are those that move a bell at most one position. How many such moves are allowed for a given $n$?

$$n = 2 \qquad (12) \quad 1 \text{ move}$$

$$n = 3 \qquad (12), (23) \quad 2 \text{ moves}$$

$$n = 4 \qquad (12), (23), (34), (12)(34) \quad 4 \text{ moves}$$

$$n = 5 \qquad 8 \text{ moves?}$$

# Change Ringing: The Number of Allowable Moves

For $n = 5$ bells, there are 7 allowable moves:

$$\underbrace{(1\ 2), (2\ 3), (3\ 4), (1\ 2)(3\ 4),}_{\text{moves from } n=4} \quad \underbrace{(1\ 2)(4\ 5), (2\ 3)(4\ 5),}_{\text{moves from } n=3} \quad \underbrace{(4\ 5).}_{\text{new move}}$$

Let $B_n$ denote the number of permissible moves on $n$ bells. We have $B_2 = 1$, $B_3 = 2$, $B_4 = 4$ and now $B_5 = 7$. Note that

$$B_5 = B_4 + B_3 + 1.$$

This equation holds in general:

$$B_n = B_{n-1} + B_{n-2} + 1$$

Where have we seen this before? Fibonacci numbers!

| $n$ | # of allowable moves | # of allowable moves $+1$ |
|---|---|---|
| 2 | 1 | 2 |
| 3 | 2 | 3 |
| 4 | 4 | 5 |
| 5 | 7 | 8 |
| 6 | 12 | 13 |
| 7 | 20 | 21 |
| 8 | 33 | 34 |
| 9 | 54 | 55 |
| 10 | 88 | 89 |
| 11 | 143 | 144 |
| 12 | 232 | 233 |

Table : The number of allowable moves for $n$ bells, where no bell moves more than one position, as a function of $n$. The right-hand column is the famous Fibonacci sequence. This shows the relation $B_n = F_{n+1} - 1$.

| $n$ | $n!$ | Approximate Duration | Name |
|---|---|---|---|
| 3 | 6 | 15 secs. | *Singles* |
| 4 | 24 | 1 mins. | *Minimus* |
| 5 | 120 | 5 mins. | *Doubles* |
| 6 | 720 | 30 mins. | *Minor* |
| 7 | 5,040 | 3 hrs. | *Triples* |
| 8 | 40,320 | 24 hrs. | *Major* |
| 9 | 362,880 | 9 days | *Caters* |
| 10 | 3,628,800 | 3 months | *Royal* |
| 11 | 39,916,800 | 3 years | *Cinques* |
| 12 | 479,001,600 | 36 years | *Maximus* |

Table : Approximate duration to ring an extent on $n$ bells and the names given to such an extent, e.g., Plain Bob Minimus, Grandshire Triples

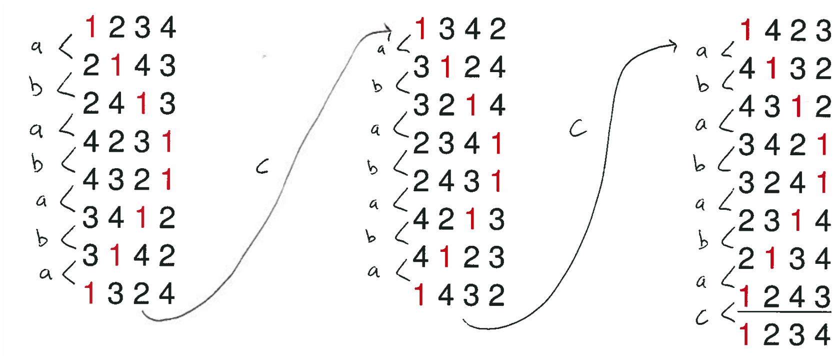
# Change Ringing: 3 bells

The two extents on 3 bells:

| | |
|---|---|
| 1 2 3 | 1 2 3 |
| 2 1 3 | 1 3 2 |
| 2 3 1 | 3 1 2 |
| 3 2 1 | 3 2 1 |
| 3 1 2 | 2 3 1 |
| 1 3 2 | 2 1 3 |
| 1 2 3 | 1 2 3 |

Note the simple zig-zag pattern of Bell 1 in the first extent, sweeping easily from position 1 to position 3 and back again. We say that Bell 1 is plain hunting; it is considered to be a non-working bell. Notice that in the second extent, Bell 1 follows a similar zig-zag path except that this begins on the second change. If $a = (1\ \ 2)$ and $b = (2\ \ 3)$, then the left extent factors as $[a\ b]^3$, while the right extent can be written as $[b\ a]^3$.

Plain Bob Minimus (read down first, then hop to next column)




Let $a = (12)(34)$, $b = (23)$, $c = (34)$. The above sequence of 24 permutations can be "factored" as

$$[(ab)^3 ac]^3 \;=\; [abababa\,c]^3 \qquad \text{Palindrome!}$$

**Note:** Plain Bob Minimus satisfies all six rules of an extent.

# Canterbury Minimus

| | | |
|---|---|---|
| 1 2 3 4 | 1 3 4 2 | 1 4 2 3 |
| 2 1 4 3 | 3 1 2 4 | 4 1 3 2 |
| 2 4 1 3 | 3 2 1 4 | 4 3 1 2 |
| 2 4 3 1 | 3 2 4 1 | 4 3 2 1 |
| 4 2 3 1 | 2 3 4 1 | 3 4 2 1 |
| 4 2 1 3 | 2 3 1 4 | 3 4 1 2 |
| 4 1 2 3 | 2 1 3 4 | 3 1 4 2 |
| 1 4 3 2 | 1 2 4 3 | 1 3 2 4 |
| | | 1 2 3 4 |

Exercise: Is this a legitimate extent? Factor the extent using $a = (1\ \ 2)(3\ \ 4)$, $b = (2\ \ 3)$, $c = (3\ \ 4)$, and $d = (1\ \ 2)$. Which of the six rules are satisfied? Compare this extent with Plain Bob Minimus.