

Symmetry in Music

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Math, Music and Identity
Montserrat Seminar Spring 2015
February 6, 11, and 13, 2015

Symmetry Operations in Music

How to get more music out of a little motif:

Translations (shifting graph vertically) \iff Transpositions (shifting notes up or down)

Example: **Stadium sports chants (organ)**

Vertical Reflection (symmetry between right and left) \iff
Retrograde (music same forward and backward)

Example: **Lean on Me**

Horizontal Reflection (symmetry between top and bottom) \iff
Inversion (what goes up, must come down)

Example: **Bach, Bach and more Bach**

Symmetry in Music: Transposition

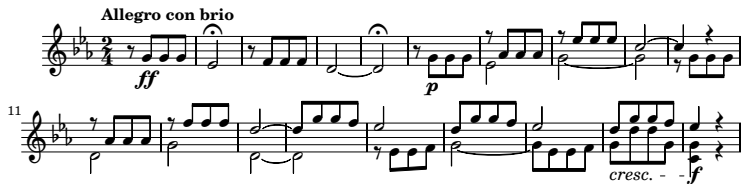


Figure : The opening measures of Beethoven's famous fifth symphony.



Figure : The hauntingly sublime opening melody of Samuel Barber's *Adagio for Strings*.

Symmetry in Music: Retrograde – Haydn

The image displays the musical score for the first 20 measures of the 'Menuetto al Rovescio' from Joseph Haydn's Piano Sonata in A Major. The score is written for piano in 3/4 time with a key signature of two sharps (F# and C#). The title 'Menuetto al Rovescio' is written above the first staff. The first system shows measures 1 through 5, with a fermata over the final note of measure 5. The second system shows measures 6 through 10, with a repeat sign at the end of measure 10. The third system shows measures 11 through 20, with a fermata over the final note of measure 20. The notation includes treble and bass staves with various note values, rests, and articulation marks.

Figure : Joseph Haydn, *Piano Sonata in A Major*, (Hob. XVI/26; Landon 41)
“Minuet in Reverse.” Both the minuet and trio are exact musical palindromes.

Symmetry in Music: Retrograde and Transposition

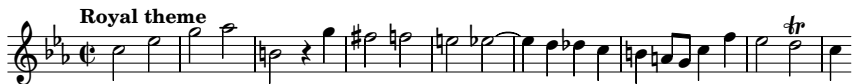
The image shows two staves of musical notation. The top staff is in treble clef with a key signature of one sharp (F#) and a common time signature (C). It contains a sequence of notes: G4 (quarter), A4 (quarter), B4 (quarter), C5 (quarter), B4-A4 (beamed eighth notes), G4 (quarter), F#4 (quarter), E4 (quarter), D4 (half). Below the staff, the lyrics "for the Lord God Om - ni - po - tent reign - eth" are written. The bottom staff is also in treble clef with a common time signature (C) and contains a sequence of notes: G4 (quarter), A4 (quarter), B4 (quarter), C5 (quarter), B4-A4 (beamed eighth notes), G4 (quarter), F#4 (quarter), E4 (quarter), D4 (half). This second staff represents a loose retrograde of the first staff, where the notes are played in reverse order from right to left.

George F. Handel, *Messiah*, *Hallelujah chorus* (loose retrograde – form of **tone painting**)

The opening minute of the piece features just two motifs, the famous “Hallelujah” motif and the excerpt above.

Johann Sebastian Bach, *Musical Offering*

- Written in 1747, three years before Bach's death, for **Frederick the Great** (King of Prussia).
- Upon visiting the King's palace, Bach was challenged by the King to improvise three-part and six-part fugues based on the "Royal theme."



- Bach succeeded in improvising a three-part fugue. Although he could not do a six-part fugue based on the Royal theme, he stunned the court audience by improvising a six-part fugue based on a theme of his own choosing.

Bach's *Musical Offering*, cont.

- Bach returned home to compose the six-part fugue, a **ricercar**, as well as several other pieces, all based on the Royal theme, and sent it to the King as his *Musical Offering*.
- The work contains 13 pieces, organized symmetrically as follows:



- A **canon** is a sophisticated type of **round**, where a main theme is imitated in some form and played by a different part after the main theme has begun. The imitations can be direct repetition; repetition at a different interval (transposition); in inversion; or in retrograde. Sometimes the theme and its imitation begin together.

Bach's *Musical Offering* – A Musical Puzzle

- Bach used all of the different symmetry types in his canons. However, to make things interesting, Bach only wrote out the full parts for one of the 10 canons. The others were left as musical puzzles, where Bach left clues to indicate how the remaining parts were to be determined.
- **Quaerendo invenietis** (“Seek and ye shall find”) was inscribed on certain canons, particularly those without titles.
- The “puzzle” offered by Bach was solved and first published by Bach’s student **Johann Philipp Kirnberger**.

Bach's *Musical Offering* – Crab Canon

Canones diversi super thema regium.**

Canon a 2.

The image shows a musical score for 'Canon a 2' from Bach's 'Musical Offering'. It consists of three staves of music. The first staff is in treble clef with a key signature of two flats (B-flat and E-flat) and a common time signature (C). The second and third staves are in bass clef with the same key signature and time signature. The music is a canon in two parts, featuring a complex rhythmic pattern and a key signature change at the very end of the piece.

Figure : The unsolved version of one of Bach's canons from the *Musical Offering*. Notice the reflected clef and key signature at the very end of the piece.

Solution for the Crab Canon

Canon a 2 (Crab Canon)

5

10

15

Figure : The *Crab Canon* from Bach's *Musical Offering*

Bach's Crab Canon – Analysis

- The primary theme (first part) consists of the Royal theme followed by an eighth-note countermelody. The entire part sounds perfectly fine in retrograde (played backwards). Thus, the second part plays the primary theme backwards **simultaneously** as the first part plays it forwards.
- Alternatively, a vertical reflection (retrograde) occurs at the end of measure nine. Each part moves in retrograde, but the parts are **interchanged**; the first part plays the second part backwards and vice-versa.
- Mathematically, this last interpretation can be visualized on a **Möbius strip!** Take the primary theme and cut it in half. Glue the two parts together, but make a twist before gluing. Each player now travels in opposite directions around the strip, with the vertical reflection taking place when the two parts pass each other after one “loop.” The twist represents the interchanging of the parts.

Bach's "Ascending" Canon

a 2. (Per tonos)*

J. P. Kirnberger

5.

Fuga canonica

*[Ascending] through the keys.

After eight measures, each part repeats, but transposed up a whole step. The ascension continues to repeat (forever?)

Bach's "Ascending" Canon cont.

Inscription by Bach for this canon: [Ascendenteque Modulatione ascendat Gloria Regis](#), or "And may the glory of the King rise with the rising modulation."

Bach poking fun at the King? (puzzles, Latin inscriptions, etc.)

A Musical Offering is a great example of Bach's ability to mix musical (and mathematical) ideas into one composition. It reveals much about Bach's musical identity and further establishes his genius as a composer and musician.

Inversions

- An **inversion** occurs when the main theme is reflected horizontally about some note. If the melody goes up by a fourth, then the inversion goes down by a fourth, etc.
- Two types of inversions: **tonal** and **exact**. A tonal inversion is one where the inversion remains in the given key; an exact inversion requires all intervals to be reflected precisely.
- For example, in the key of C major, a melody that begins on a C and goes up a major third to E, would be reflected in a tonal inversion about C to the notes C and A (down a **minor** third), in order to avoid any accidentals. If the inversion were exact, then it would be C to $A\flat$ (down a major third).



Figure : A simple melody along with its tonal and exact inversions. Here the horizontal reflection is about C, as can be viewed clearly in the lower-right excerpt.

Excerpt	Sequence of Intervals
Original Melody	↑ P4, ↓ m3, ↑ whole step, ↑ m3, ↓ P5
Exact Inversion	↓ P4, ↑ m3, ↓ whole step, ↓ m3, ↑ P5
Tonal Inversion	↓ P4, ↑ M3, ↓ whole step, ↓ M3, ↑ P5

Table : The interval sequences for the excerpts in the above figure.

Symmetry in Music: Inversion – Bartók

Allegro



Béla Bartók, *Mikrokosmos*, No. 141, *Subject and Reflection* – **exact inversion** about B \flat .

Allegro



Symmetry in Music: Inversion – Sousa

The image shows a musical score for the opening of the march 'The Thunderer' by John Philip Sousa. The score is written for piano and consists of two staves: a treble clef staff and a bass clef staff. The key signature is one flat (B-flat), and the time signature is common time (C). The music begins with a half note chord in the treble staff (F2, B-flat1) and a half note chord in the bass staff (F2, B-flat1). The melody in the treble staff starts with a half note G2, followed by a quarter note A2, and then a quarter note B-flat2. The bass staff provides a harmonic accompaniment with chords and moving lines. The piece concludes with a final chord in both staves.

John Philip Sousa, opening of the march *The Thunderer*. (Analyze for HW.)

Symmetry in Music: Inversion – Bach

The image displays three musical staves in treble clef, all in the key of D minor (three sharps: F#, C#, G#) and common time (C). The first staff, labeled "Subject", shows the original melodic line. The second staff, labeled "Inverted subject" and starting at measure 45, shows the subject inverted (mirrored across a horizontal axis), with some notes marked with an 'x' to indicate alterations. The third staff, also labeled "Inverted subject" and starting at measure 30, shows another inversion of the subject, demonstrating a different symmetrical transformation.

Figure : The subject and two different inversions of the subject in Bach's *Fugue No. 8 in D \sharp minor* from the *Well-Tempered Clavier*, vol. I

Bach: The Well-Tempered Clavier, Fugue No. 8 in D \sharp minor

Handbook for Keyboard Teacher + Performer Chapter Five A Proposed Course of Study and Analysis
by Laurette Adberg Example 5.2 Fugue No. 8 in d \sharp minor from WTC I by J.S. Bach ABA' form (Sonata-form)

The image displays a musical score for Fugue No. 8 in D \sharp minor, BWV 855, from the Well-Tempered Clavier, Part I by Johann Sebastian Bach. The score is presented in a three-staff format (treble, alto, and bass clefs). The key signature is two sharps (F \sharp and C \sharp), and the time signature is 4/4. The score is divided into sections: Exposition (measures 1-11), Tonal answer (measures 12-15), and Development (measures 16-35). The exposition begins with a treble clef and a key signature of two sharps. The first staff shows the treble clef, the second staff shows the alto clef, and the third staff shows the bass clef. The score includes various annotations such as '3' above measure 12, '8' above measure 15, '12a' above measure 16, 'S (redundant statement)' below measure 15, 'Development' above measure 16, '(rhythmic sync.)' above measures 20 and 24, and '24' above measure 24. The score ends with a double bar line and a repeat sign.

60 ^{6a} **Recapitulation**

chrom. variation

S-augmented

64 S-inverted F# Major S-aug.

69 S

73 Coda (stretto) S-aug. S

79 rhythmic orn. (orn.) S (orn.)

83

Symmetry in Music: Retrograde-inversion

Opening of Praeludium

Closing of Postludium

The image displays two musical staves from Paul Hindemith's *Ludus Tonalis*. The top staff, titled "Opening of Praeludium", shows a treble clef with a common time signature. It begins with a fortissimo (*ff*) dynamic and a triplet of eighth notes. This is followed by a sixteenth-note scale in the right hand, while the left hand remains silent. The piece concludes with a triplet of eighth notes and a fermata. The bottom staff, titled "Closing of Postludium", shows a bass clef with a common time signature. It begins with a fermata, followed by a sixteenth-note scale in the left hand, while the right hand remains silent. The piece concludes with a triplet of eighth notes and a fermata. The two sections are exact retrograde-inversions of each other.

Figure : Paul Hindemith, *Ludus Tonalis* ("Tonal Game"), beginning and end. The ending Postludium is an **exact** retrograde-inversion (180° rotation) of the opening Praeludium.

Combining Symmetries – Liszt

R.h. 8va throughout

The image displays a musical score for Franz Liszt's Hungarian Rhapsody #2, specifically the right-hand part (R.h.) with an 8va (octave) marking throughout. The score is presented in four systems, each consisting of a treble clef staff and a bass clef staff. The music is written in a complex, rhythmic style characteristic of Liszt's style, featuring many beamed notes and dynamic markings. The key signature is one sharp (F#), and the time signature is 2/4. The notation includes various ornaments and slurs, indicating a highly technical and expressive piece.

Figure : Franz Liszt, excerpt from *Hungarian Rhapsody #2*

Combining Symmetries – Gershwin

9a.

1 2 3 4 2 2 3 4 3 2 3 4 4 2 3 4

B \flat B \flat 6 C m 7 F7 B \flat E $^{\circ}$ 7 C m 7

"I Got Rhythm" has an AABA structure, and a two-bar tag at the end. We call these four equal sections A, A, B, and A. Three of the sections are the same, and one is different. Listen to the first section, and you'll be able

9b.

[A] B \flat B \flat 6 C m 7 F7 B \flat 6 E $^{\circ}$ 7 C m 7 F7 B \flat B \flat 6 C m 7 F7 E b m6

[A] B \flat F7 B \flat B \flat B \flat 6 C m 7 F7 B \flat 6 E $^{\circ}$ 7 C m 7 F7 B \flat B \flat 6

[B] Bridge C m 7 F7 E b m6 B \flat F7 B \flat D7 A m 7 F m 6 D7 G D+ G9 G7

[A] C7 G m 7 E b m6 C9 C7-5 F7 B \flat B \flat 6 C m 7 F7 B \flat 6 E $^{\circ}$ 7

Tag C m 7 F7 B \flat B \flat 6 C m 7 F7 E b m B \flat F m G7 C7 F7 B \flat

Figure : George Gershwin, *I Got Rhythm*, (transposition, retrograde and inversion, all in one song!)

Der Spiegel (The Mirror) Duet

VOLTA 1 *Allegro* $J=120$

W.A. Mozart

mf

Allegro

Public Domain. Synchronized by Fred Nachbauer using NoteWorthy
Codemore? Try playing this from opposite sides of a table.

(Note: the attribution to Mozart is dubious)

Figure 9.6.