Math, Music and Identity

Worksheet: Change Ringing

Rules for ringing an EXTENT on n bells:

1. The first and last changes (rows) are *rounds* ($1\ 2\ 3\ 4 \cdots n$).
2. Other than rounds, all of the other $n!$ changes occur exactly once.
3. Between successive changes, no bell moves more than one position (no jumping).
4. No bell rests for more than 2 (sometimes relaxed further to 4) changes.
5. Each *working bell* should do the same amount of "work" (obey the same overall pattern).
6. Horizontal symmetry should be present in the extent to help the ringers learn the path of their respective bell. This is called the *palindrome property*.

Note: Rules 1 - 3 are mandatory for an extent while Rules 4 - 6 are optional though often satisfied.

n	$n!$	Approximate Duration	Name
3	6	15 secs.	*Singles*
4	24	1 mins.	*Minimus*
5	120	5 mins.	*Doubles*
6	720	30 mins.	*Minor*
7	5,040	3 hrs.	*Triples*
8	40,320	24 hrs.	*Major*
9	362,880	9 days	*Caters*
10	3,628,800	3 months	*Royal*
11	39,916,800	3 years	*Cinques*
12	479,001,600	36 years	*Maximus*

Table 1: The approximate durations to ring an extent on n bells covering all $n!$ changes. The titles given to extents on n bells are also displayed. For example, *Plain Bob Minimus* is a composition written for 4 bells, while *Grandshire Triples* is an extent on 7 bells.

Let B_n denote the number of permissible moves on n bells: $B_2 = 1, B_3 = 2, B_4 = 4, B_5 = 7, \ldots$ Then we have the recursive relation

$$B_n = B_{n-1} + B_{n-2} + 1, \qquad \text{which implies that}$$

$$\boxed{\text{The \# of permissible moves on } n \text{ bells is one less than the } (n+1)\text{th Fibonacci number.}}$$

The two extents on 3 bells:

1 2 3	**1** 2 3
2 **1** 3	**1** 3 2
2 3 **1**	3 **1** 2
3 2 **1**	3 2 **1**
3 **1** 2	2 3 **1**
1 3 2	2 **1** 3
1 2 3	**1** 2 3

Note the simple zig-zag pattern of Bell **1** in the first extent, sweeping easily from position 1 to position 3 and back again. We say that Bell **1** is *plain hunting*; it is considered to be a "non-working" bell. Notice that in the second extent, Bell **1** follows a similar zig-zag path except that this begins on the second change. If $a = (1\ 2)$ and $b = (2\ 3)$, then the left extent factors as $[a\ b]^3$, while the right extent can be written as $[b\ a]^3$.

Plain Bob Minimus (read down first, then hop to next column)

1 2 3 4	**1** 3 4 2	**1** 4 2 3
2 **1** 4 3	3 **1** 2 4	4 **1** 3 2
2 4 **1** 3	3 2 **1** 4	4 3 **1** 2
4 2 3 **1**	2 3 4 **1**	3 4 2 **1**
4 3 2 **1**	2 4 3 **1**	3 2 4 **1**
3 4 **1** 2	4 2 **1** 3	2 3 **1** 4
3 **1** 4 2	4 **1** 2 3	2 **1** 3 4
1 3 2 4	**1** 4 3 2	<u>**1** 2 4 3</u>
		1 2 3 4

Note the similarities with the first extent on three bells above. Bell **1** goes plain hunting again, this time moving from position 1 to position 4 and back again, needing 3 cycles to complete the extent. The other three bells have the same paths, just starting at different places so that Rule 5 is satisfied. This is a bit like a round. Letting $a = (1\,2)(3\,4), b = (2\,3)$ and $c = (3\,4)$ be the three permutations used in the extent, *Plain Bob Minimus* can be "factored" as $[(ab)^3ac]^3$. Since *abababa* is a palindrome, Rule 6 is satisfied for this extent. There are lots of interesting patterns and mathematics lurking in this extent!

Canterbury Minimus (read down first, then hop to next column)

1 2 3 4	**1** 3 4 2	**1** 4 2 3
2 **1** 4 3	3 **1** 2 4	4 **1** 3 2
2 4 **1** 3	3 2 **1** 4	4 3 **1** 2
2 4 3 **1**	3 2 4 **1**	4 3 2 **1**
4 2 3 **1**	2 3 4 **1**	3 4 2 **1**
4 2 **1** 3	2 3 **1** 4	3 4 **1** 2
4 **1** 2 3	2 **1** 3 4	3 **1** 4 2
1 4 3 2	**1** 2 4 3	<u>**1** 3 2 4</u>
		1 2 3 4

Is this a legitimate extent? Factor the set of changes and determine whether Rules 4, 5, or 6 are satisfied. You will need an additional move: $d = (1\ 2)$. Compare this extent with *Plain Bob*.

Group Theory Revisited

Consider the extent *Plain Bob Minimus* as a list of all permutations in the group S_4.

$e =$	**1** 2 3 4	**1** 3 4 2	**1** 4 2 3
$\alpha =$	2 **1** 4 3	3 **1** 2 4	4 **1** 3 2
$\beta =$	2 4 **1** 3	3 2 **1** 4	4 3 **1** 2
$\alpha\beta =$	4 2 3 **1**	2 3 4 **1**	3 4 2 **1**
$\beta^2 =$	4 3 2 **1**	2 4 3 **1**	3 2 4 **1**
$\alpha\beta^2 =$	3 4 **1** 2	4 2 **1** 3	2 3 **1** 4
$\beta^3 =$	3 **1** 4 2	4 **1** 2 3	2 **1** 3 4
$\beta\alpha =$	**1** 3 2 4	**1** 4 3 2	<u>**1** 2 4 3</u>
			1 2 3 4

We label the first two changes after rounds as α and β (α and β are pronounced "alpha" and "beta," respectively. These are the first two letters of the Greek alphabet, commonly used in mathematics.) One can check that the remaining five changes of the first lead are all expressible in terms of α and β, given by the formulas shown above. Note that we are dropping the $*$ here for ease of notation, so for example, $\alpha\beta = \alpha * \beta$. Remember that $\alpha\beta$ means we apply the permutation α first, then take the result and apply β. The order matters!

The goal of the remainder of this worksheet is to prove that the first column (lead) of *Plain Bob Minimus* forms a group under multiplication of permutations. Let $K = \{e, \beta, \beta^2, \beta^3, \alpha, \alpha\beta^2, \alpha\beta, \beta\alpha\}$. We will show that K is a subgroup of the larger group S_4. The hardest thing to show is that the elements in K are closed under multiplication of permutations. In other words, the product of any two permutations in K remains in K. This can be accomplished by making a multiplication table and checking that all 64 products are indeed elements of the first lead.

Instead of doing every multiplication out by hand, there are some identities involving α and β that are particularly useful. In turn, these identities can be used to derive other useful relations. You might keep a list of them as you fill out your table. Here are some key ones:

$$\begin{aligned} \alpha^2 &= e \\ \beta^4 &= e \\ \beta\alpha\beta &= \alpha \end{aligned}$$

Using the Identities

Suppose we wanted to find the product $\alpha\beta^3$ using the identities above. Using the last identity, and multiplying **on the right** of each side by β^3, gives us

$$\beta\alpha\beta * \beta^3 = \alpha * \beta^3$$

The left-hand side of this equation simplifies to $\beta\alpha\beta^4 = \beta\alpha e = \beta\alpha$, which shows that $\alpha\beta^3 = \beta\alpha$. It is very important that you multiply the same way on each side of the equation. Since $*$ is not usually commutative, the order matters! Fill out the multiplication table on the next page to show that K is closed under multiplication of permutations. A few rows have been completed to help get you started.

$*$	e	β	β^2	β^3	α	$\alpha\beta^2$	$\alpha\beta$	$\beta\alpha$
e	e	β	β^2	β^3	α	$\alpha\beta^2$	$\alpha\beta$	$\beta\alpha$
β								
β^2								
β^3								
α	α	$\alpha\beta$	$\alpha\beta^2$	$\beta\alpha$	e	β^2	β	β^3
$\alpha\beta^2$								
$\alpha\beta$								
$\beta\alpha$								

Table 2: The multiplication table for the eight permutations in the first lead of *Plain Bob Minimus*, denoted as the subgroup K. Complete the table to show that K is closed. What is the inverse of each element in K? Conclude that K is a group in its own right, a subgroup of S_4. How does this table compare with the table you constructed for the dihedral group D_4, the symmetries of the square?

Ringing the Cosets: A Mathematical Tangent

It is tempting to ask whether the other leads in *Plain Bob Minimus* are also subgroups. The answer is no since neither contains rounds, which is the identity element, so property 3 for groups does not hold. However, it is easy to generate these leads from the subgroup formed by the first lead. Multiplication by the permutation (1 3 4 2) on the right takes the entire first column to the second column and multiplication again by this same permutation takes the second column to the third.

In group theory, the second and third leads are called **cosets**. A coset is obtained from a subgroup by multiplying on the left or the right every element in the subgroup. Thus we can speak of left or right cosets. Each coset has the same number of elements as the subgroup just as our three leads each have the same number of changes (eight). It turns out that many extents have this decomposition where the first lead is a subgroup and the remaining leads are just cosets generated by this group. Mathematician and composer Arthur White acknowledges this fact in the title of his wonderful paper on change ringing, *Ringing the Cosets*, American Mathematical Monthly, Oct. 1987. Moreover, the elements used in generating the other leads form a different subgroup together, called a **cyclic subgroup of order** n. This is a group of the form $\{\omega, \omega^2, \omega^3, \ldots, \omega^n = e\}$. It is clear that there is a great deal of group theory involved in doing change ringing! See Section 6.2 of the course text for more information on this topic.