A Continuum of Relative Equilibria in the 5-Body Problem

Gareth E. Roberts
Math. Dept. Boston Univ.
111 Cummington Street
Boston, MA 02215
garethr@math.bu.edu
(617) 353-1484

November 18, 1998

Abstract

It is generally believed that the set of relative equilibria equivalence classes in the Newtonian n-body problem, for a given set of positive masses, is finite. However, the result has only been proven for $n = 3$ and remains a difficult, open question for $n \geq 4$ (Wintner [16], Smale [14]). We demonstrate that the condition for the masses being positive is a necessary one by finding a continuum of relative equilibria in the five-body problem which (unfortunately) includes one negative mass. This family persists in similar potential functions, including the logarithmic potential used to describe the motion of point vortices in a plane of fluid.

Keywords: n-body problem, relative equilibria
MSC Codes: 70F10, 70F15

1 Introduction

A relative equilibrium is a special configuration of masses of the n-body problem which rotates rigidly about its center of mass if given the correct initial momenta. In rotating coordinates these special solutions become fixed points, hence the name *relative equilibria.* These periodic orbits are the only explicitly known solutions of the n-body problem.

A natural question, posed by Wintner in [16], is whether the number of relative equilibria is finite for a given set of positive masses. For $n = 3$, there are only five relative equilibria for any set of masses: the three collinear solutions of Euler [3] and the two equilateral solutions of Lagrange [5]. For $n = 4$, Albouy recently classified all the relative equilibria for equal masses [1]. There are essentially only four configurations (50 if you include all possible permutations of the masses): a collinear solution, a square, an equilateral triangle with a body at the center and an isosceles triangle with a body on the axis of symmetry. For the collinear case, there are exactly $n!/2$ relative equilibria [10]. While many examples of relative equilibria have been found for larger n, no complete classification exists for four unequal masses or for $n \geq 5$. In fact, Smale poses Wintner's question as one of his problems for the 21st century (see [14]).

In this note, we show the existence of a one-parameter family of degenerate relative equilibria for the 5-body problem which contains a negative mass (ie. repulsive force). This family contains a degeneracy which does not arise from rotation or scaling. Due to the negative mass, it is not a counterexample to the Wintner/Smale question, but does demonstrate the necessity of having positive masses. Moreover, relative equilibria can be defined for other potential functions and in some of these cases, "negative masses" are physically reasonable. (The case with charged particles is more complicated and is not being alluded to here.) If we generalize the Wintner/Smale question to other potential functions, including the logarithmic potential for the motion of point vortices (often referred to as the Kirchhoff problem [4], [11]), this family persists. Although having a negative mass in the gravitational n-body problem is not physically realistic, having a negative circulation in the Kirchhoff problem is physically reasonable.

2 Relative Equilibria

We let the mass and position of the n bodies be given by m_i and $\mathbf{q}_i \in \mathbb{R}^2$, $i = 1, \ldots, n$. Let $r_{ij} = \|\mathbf{q}_i - \mathbf{q}_j\|$ be the distance between the ith and jth bodies and let $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_n) \in \mathbb{R}^{2n}$. Using Newton's law of motion and the inverse square law for attraction due to gravity, the second-order equation for the ith body is given by

$$m_i \ddot{\mathbf{q}}_i = \sum_{i \neq j} \frac{m_i m_j (\mathbf{q}_j - \mathbf{q}_i)}{r_{ij}^3} = \frac{\partial U}{\partial \mathbf{q}_i},$$

where $U(\mathbf{q})$ is the Newtonian potential function:

$$U(\mathbf{q}) = \sum_{i<j} \frac{m_i m_j}{r_{ij}}.$$

We let the momenta of each body be $\mathbf{p}_i = m_i \dot{\mathbf{q}}_i$ and let $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n) \in \mathbb{R}^{2n}$. The equations of motion can then be written as

$$\begin{aligned} \dot{\mathbf{q}} &= M^{-1}\mathbf{p} \\ \dot{\mathbf{p}} &= \nabla U(\mathbf{q}) \end{aligned} \tag{1}$$

where M is the diagonal mass matrix with diagonal $m_1, m_1, m_2, m_2, \ldots, m_n, m_n$.

A *relative equilibrium* of the n-body problem is a configuration $\mathbf{x} \in \mathbb{R}^{2n}$ which satisfies the algebraic equations

$$\nabla U(\mathbf{x}) + \omega^2 M\mathbf{x} = 0 \tag{2}$$

for some value of ω (see [6], [7], [9], or [15] for details). The ith component in (2) is given by

$$-\omega^2 m_i \mathbf{x}_i = \sum_{i \neq j} \frac{m_i m_j (\mathbf{x}_j - \mathbf{x}_i)}{r_{ij}^3}.$$

Note that if $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ is a relative equilibrium, then

$$\begin{aligned} c\mathbf{x} &= (c\mathbf{x}_1, c\mathbf{x}_2, \ldots, c\mathbf{x}_n) \quad \text{and} \\ R\mathbf{x} &= (R\mathbf{x}_1, R\mathbf{x}_2, \ldots, R\mathbf{x}_n) \end{aligned}$$

are also relative equilibria for any constant c and any $R \in SO(2)$. When counting relative equilibria it is standard to fix the size (a unique value of c) and identify any configurations which are rotationally equivalent via the equivalence relation $\mathbf{x} \sim R\mathbf{x}$ for $R \in SO(2)$. Indeed, if we let $I(\mathbf{x})$ denote the moment of inertia, that is,

$$I(\mathbf{x}) = \frac{1}{2} \sum_{i=1}^{n} m_i \|\mathbf{x}_i\|^2,$$

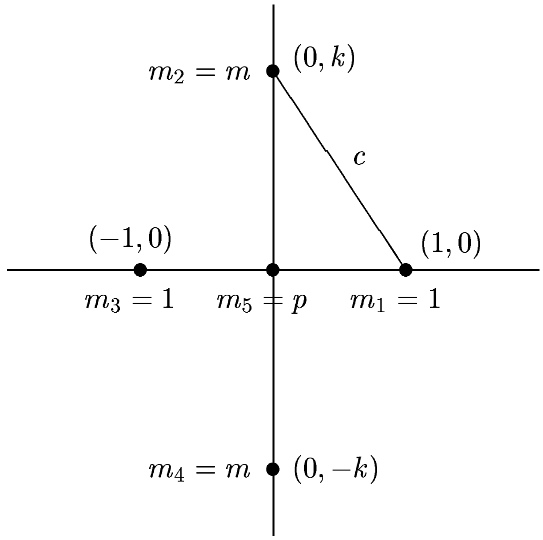

Figure 1: Set up for the 1+rhombus relative equilibria.

we can then write equation (2) as

$$\nabla U(\mathbf{x}) + \omega^2 \nabla I(\mathbf{x}) = 0.$$

In other words, relative equilibria are critical points of the function $U([\mathbf{x}])$ restricted to the mass ellipsoid $I = c$, where $[\mathbf{x}]$ is the equivalence class of $\mathbf{x}$ and ω^2 plays the role of a Lagrange multiplier. Thus any question on the number of relative equilibria is really a question on the number of critical points of $U([\mathbf{x}])$ restricted to an inertia manifold. If this number were not finite, then a continuum of relative equilibria would exist [12]. (Here we mean a continuum other than one arising from scaling or rotation.) In other words, is it possible to find a continuum of critical points of $U([\mathbf{x}])$ restricted to the mass ellipsoid $I = c$? While we cannot answer this question in the affirmative, we do find such a continuum, but containing a negative mass.

3 1+Rhombus Relative Equilibria

The easiest and most accessible relative equilibria are those configurations which contain large amounts of symmetry. For example, placing n equal masses at the vertices of a regular n-gon and adding a body of mass m at its center is a relative equilibria for all values of the central mass m.

We begin our search for a family of degenerate relative equilibria by considering a configuration which consists of four bodies at the vertices of a rhombus, with opposite vertices having the same mass, and a central body. There will be three parameters in this family: the shape of the rhombus, the mass ratio of the bodies on the rhombus and the mass of the central body.

We let $m_1 = 1, m_2 = m, m_3 = 1, m_4 = m, m_5 = p$ be the five masses and $\mathbf{q}_1 = (1, 0), \mathbf{q}_2 = (0, k), \mathbf{q}_3 = (-1, 0), \mathbf{q}_4 = (0, -k), \mathbf{q}_5 = (0, 0)$ be the positions of the five bodies. We will let $c = \sqrt{k^2 + 1}$ be the side of the rhombus (see Figure 1).

Due to the symmetry, the forces on the first and third bodies differ only in sign, as do the forces on the second and fourth bodies. The forces on the central body cancel out to zero. Equation (2) then reduces to the two equations

$$\frac{m}{c^3}\begin{pmatrix} -1 \\ k \end{pmatrix} + \frac{1}{8}\begin{pmatrix} -2 \\ 0 \end{pmatrix} + \frac{m}{c^3}\begin{pmatrix} -1 \\ -k \end{pmatrix} + p\begin{pmatrix} -1 \\ 0 \end{pmatrix} + \omega^2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\frac{1}{c^3}\begin{pmatrix} 1 \\ -k \end{pmatrix} + \frac{1}{c^3}\begin{pmatrix} -1 \\ -k \end{pmatrix} + \frac{m}{(2k)^3}\begin{pmatrix} 0 \\ -2k \end{pmatrix} + \frac{p}{k^3}\begin{pmatrix} 0 \\ -k \end{pmatrix} + \omega^2\begin{pmatrix} 0 \\ k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which are satisfied when $\omega^2 = 2m/c^3 + 1/4 + p$ and

$$m(\frac{2}{c^3} - \frac{1}{4k^3}) + p(1 - \frac{1}{k^3}) = \frac{2}{c^3} - \frac{1}{4}. \tag{3}$$

For each k-value, equation (3) determines a linear relationship between m and p. In other words, fixing the size of the rhombus yields a one-parameter family of relative equilibria for which the masses m and p change linearly with respect to each other. Alternatively, we can ask whether a given pair (m, p) has more than one k-value associated to it for which equation (3) is satisfied. This question is surprisingly answered in the affirmative. The pair $(m = 1, p = -1/4)$ makes equation (3) degenerate. Thus we have found a one-parameter family of degenerate relative equilibria for the fixed set of masses $(1, 1, 1, 1, -1/4)$ in the five-body problem. Physically, the negative central mass acts as a repeller cancelling out the forces along both axes and the corresponding centrifugal force needed to maintain the equilibria works out to be the same for the 1st and 3rd bodies as it is for the 2nd and 4th.

For an alternative view, let $\mathbf{x}_1 = (j, 0)$ and $\mathbf{x}_3 = (-j, 0)$, instead of fixing them one unit distance away from the origin, and let the other positions remain as given above. Let the masses be the special values $(1, 1, 1, 1, -1/4)$. Then $c = \sqrt{j^2 + k^2}$ and $I = j^2 + k^2$ so that fixing $I = 1$ implies

$$U = 4 \cdot 1 + \frac{1}{2j} + \frac{1}{2k} - \frac{1}{4}(\frac{2}{j} + \frac{2}{k}) = 4.$$

Therefore, for the masses $(1, 1, 1, 1, -1/4)$, the potential function U restricted to the ellipsoid $I = 1$ has a curve of critical points at

$$(\cos t, 0, 0, \sin t, -\cos t, 0, 0, -\sin t, 0, 0) \quad \text{for} \quad 0 \leq t \leq \pi/2.$$

Note that this continuum has endpoints located at points on the ellipsoid where three of the five bodies collide. This does not happen in the classical problem (with positive masses) as relative equilibria cannot accumulate on collision points (see [2] or [13]). Here, the potential function U remains constant (and particularly bounded) as you approach the collision points on the mass ellipsoid because the negative mass cancels out the forces felt by the three masses regardless of their distances apart.

Theorem 3.1 *In the five-body problem for the masses* $(1, 1, 1, 1, -1/4)$*, there exists a one-parameter family of degenerate relative equilibria where the four equal masses are positioned at the vertices of a rhombus with the remaining body located at the center. As the parameter varies, one pair of opposite vertices move away from each other while the other pair moves closer, maintaining a fixed length between consecutive vertices.*

Corollary 3.2 *The number of relative equilibria equivalence classes in the five-body problem for the masses* $(1, 1, 1, 1, -1/4)$ *is* **not** *finite.*

4 Concluding Remarks

1. A calculation similar to the one above shows that this family of relative equilibria persists in similar potential functions, including the logarithmic potential used in the Kirchhoff problem. Specifically, if we let $U_0 = -\Sigma m_i m_j \ln r_{ij}$ and $U_d = \Sigma m_i m_j / r_{ij}^d$ for $d > 0$, then the 1+ rhombus family described above is a one-parameter family of degenerate relative equilibria for these potential functions also, with the only difference being that the central mass is given by $p = -1/2^{d+1}$. Generalizing Wintner's question, we see that the number of relative equilibria equivalence classes in the five-body problem is also **not** finite for these other potential functions.

2. A natural attempt for modifying our example to obtain another (positive mass) family of degenerate relative equilibria is to replace the negative mass at the origin with four symmetrically located bodies with positive masses about a positive mass in the center. While this does not lead to the discovery of a continuum, it does produce a surprising one-parameter family of relative equilibria in the 9-body problem. This family consists of four bodies of mass one located at the vertices of a square of radius one, four bodies of *arbitrary* mass m located at the vertices of a square of radius $j \approx .5318$ (aligned with the outer square), and a body at the origin of mass $p \approx 1.3022$. What makes this family unusual is that the mass of the four equal bodies on the inner square serves as the parameter, in contrast to many other families of relative equilibria (for example, the regular n-gon with a central mass, see [7] or [8]), where the mass of the central body acts as the parameter.

Acknowledgments: I would like to thank Glen Hall and Alain Albouy for their advice and suggestions regarding this work.

References

[1] A. Albouy. The symmetric central configurations of four equal masses. *Hamiltonian dynamics and celestial mechanics (Seattle, WA, 1995)*, 131-135, Contemp. Math., **198**, 1996.

[2] G. Buck. On clustering in central configurations. *Proceedings of the Amer. Math. Soc.* **108**, no. 3, 801-810, 1990.

[3] L. Euler. De motu rectilineo trium corporum se mutuo attahentium. *Novi Comm. Acad. Sci. Imp. Petrop.* **11**, 144-151, 1767.

[4] G. R. Kirchhoff. *Vorlesungen uber Matematische Physik*, Vol. I, Teubner, Leipzig, 1876.

[5] J. L. Lagrange. Essai sur le problème des trois corps. *Œuvres, vol. 6*, 272-292, Gauthier-Villars, Paris, 1772.

[6] K. Meyer and G. R. Hall. *Introduction to Hamiltonian Dynamical Systems and the N-Body Problem*, Applied Mathematical Sciences, 90. Springer, New York, 1992.

[7] K. Meyer and D. S. Schmidt. Bifurcations of Relative Equilibria in the N-Body and Kirchoff Problems. *SIAM J. Math. Anal.* **19**, no. 6, 1295-1313, 1988.

[8] R. Moeckel. Linear Stability Analysis of Some Symmetrical Classes of Relative Equilibria. *Hamiltonian dynamical systems (Cincinnati, OH, 1992)*, 291-317, IMA Vol. Math Appl., 63. Springer, New York, 1995.

[9] R. Moeckel. Relative equilibria of the four-body problem. *Ergodic Theory and Dynamical Systems* **5**, 417-435, 1985

[10] F. R. Moulton. The straight line solutions of the n-body problem. *Annals of Math.*, II. Ser. **12**, 1-17, 1910.

[11] K. O'Neil. Stationary Configurations of Point Vortices. *Trans. Amer. Math. Soc.* **302**, no. 2, 383-425, 1987.

[12] J. Palmore. New Relative Equilibria of the N-Body Problem. *Letters in Mathematical Physics.* **1**, no. 2, 119-123, 1976.

[13] M. Shub. Diagonals and relative equilibria. *Manifolds-Amsterdam* 1970 *Lecture Notes in Math.*, **197**, Springer, Berlin, 1971.

[14] S. Smale. Mathematical Problems for the Next Century. *Mathematical Intelligencer* **20**, 7-15, Spring 1998.

[15] S. Smale. Topology and mechanics, II, The planar n-body problem. *Invent. Math.* **11**, 45-64, 1970.

[16] A. Wintner. The Analytical Foundations of Celestial Mechanics. *Princeton Math. Series* **5**, 215, Princeton University Press, Princeton, NJ, 1941.