Linear Stability Analysis of the Figure-eight Orbit in the Three-body Problem

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Abstract

We show that the well-known figure-eight orbit of the three-body problem is linearly stable. Building on the strong amount of symmetry present, the monodromy matrix for the figure-eight is factored so that its stability can be determined from the first twelfth of the orbit. Using a clever change of coordinates, the problem is then reduced to a 2×2 matrix whose entries depend on solutions of the associated linear differential system. These entries are estimated rigorously using only a few steps of a Runge-Kutta-Fehlberg algorithm. From this we conclude that the characteristic multipliers are distinct and lie on the unit circle. The methods and results presented are applicable to a wide range of Hamiltonian systems containing symmetric periodic solutions.

MSC Classifications: 70F10, 70F15

Key Words: n-body problem, linear stability, figure-eight orbit, symmetric periodic solutions

1 Introduction

In December of 1999, at the International Conference on Celestial Mechanics dedicated to Don Saari, Alain Chenciner and Richard Montgomery announced a wonderful variational proof of a fascinating periodic solution in the classical Newtonian 3-body problem [9]. The *figure-eight orbit* consists of three equal mass particles traveling around a fixed figure-eight curve in the plane, spaced apart by equal time intervals. This special orbit was first discovered numerically by Cristopher Moore [17]. Chenciner and Montgomery showed that the action minimizer ζ over the class of curves traveling from an Euler central configuration to an isosceles triangle does not contain any collisions. They construct the full figure-eight orbit by gluing 12 copies of ζ together smoothly via rotations and reflections (see Figure 1.) The success of their approach launched a flurry of activity in the field leading to many more special periodic solutions of the n-body problem (see [4, 5, 7, 8, 11, 21] for example, and their references.)

One such class of solutions, which includes the figure-eight, consists of orbits where all n bodies follow each other around a closed curve with equally spaced time gaps. These special solutions have become known as *choreographies*. In [21], Carles Simó numerically located hundreds of such orbits but distinguishes the figure-eight, stating that, "All the choreographies found, except the eight, are unstable." Extensive numerical calculations claiming both linear and nonlinear stability of the figure-eight were given by Simó at the same conference where Chenciner and Montgomery first described the eight [20]. Similar but less detailed results were later announced in [12].

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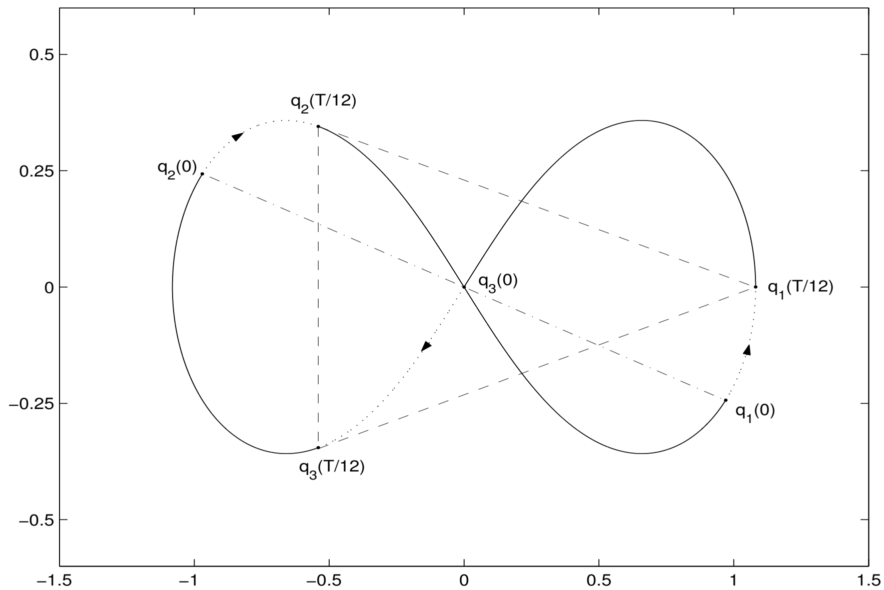

Figure 1: The first twelfth of the figure-eight orbit (dotted), traveling from an Euler collinear central configuration to an isosceles triangle. The entire figure-eight can be constructed by reflecting the first twelfth about each coordinate axis.

In this paper we focus on an analytic argument for linear stability of the figure-eight orbit. This is challenging since no closed form expression for the orbit exists. Moreover, the analytic calculation of Floquet multipliers is difficult even when the coefficients of the corresponding linear system are known. We use the extensive symmetries of the figure-eight solution to reduce the stability calculations down to the behavior of the first twelfth of the orbit. These techniques are first described for generic Hamiltonian systems possessing a certain type of symmetric periodic solutions. We then make a special change of coordinates to reduce the problem further to a 2×2 matrix. At this point we are forced to do a minimal amount of numerical work, using just four steps of a Runge-Kutta-Fehlberg method to rigorously place the multipliers on the unit circle. The values obtained not only show the linear stability of the figure-eight but also verify that it is nondegenerate and may be continued as a periodic solution with less symmetry [10].

Further work has continued on this important problem since the current paper was submitted. In [14], Kapela and Simó use multiple precision interval arithmetic to numerically prove linear stability of the eight. Their work involves rigorous estimates of the full monodromy matrix although they do not use shape sphere coordinates nor the available reductions due to symmetry utilized here.

1.1 The figure-eight orbit

The Newtonian n-body problem concerns the motion of n point masses interacting solely under their gravitational attraction. Denote m_i and $\mathbf{q}_i \in \mathbb{R}^2$ as the mass and position, respectively, of the ith particle. The gravitational force is determined by the gradient of the self-potential

$$U(\mathbf{q}) = \sum_{i<j} \frac{m_i m_j}{r_{ij}}.$$

The equations of motion for the ith body are

$$m_i \ddot{\mathbf{q}}_i = \frac{\partial U}{\partial \mathbf{q}_i} = \sum_{j \neq i} \frac{m_i m_j (\mathbf{q}_j - \mathbf{q}_i)}{r_{ij}^3}$$

where $r_{ij} = ||\mathbf{q}_i - \mathbf{q}_j||$ measures the distance between the ith and jth bodies. Without loss of generality we assume that the center of mass $\bar{\mathbf{q}} = \frac{1}{M} \sum_{i=1}^n m_i \mathbf{q}_i$ is always at the origin. The moment of inertia I and kinetic energy K are given by

$$I(\mathbf{q}) = \sum_{i=1}^n m_i \|\mathbf{q}_i\|^2 \quad \text{and} \quad K(\dot{\mathbf{q}}) = \frac{1}{2} \sum_{i=1}^n m_i ||\dot{\mathbf{q}}_i||^2,$$

while the Hamiltonian governing the equations of motion is $H(\mathbf{q}, \dot{\mathbf{q}}) = K(\dot{\mathbf{q}}) - U(\mathbf{q})$.

Definition 1.1 *A* **choreography** *is a T-periodic solution of the n-body problem where all bodies follow the same loop $q(t)$ with equal time spacing. In other words,*

$$\gamma(t) = (q(t + \tfrac{n-1}{n}T), q(t + \tfrac{n-2}{n}T), \cdots, q(t + \tfrac{T}{n}), q(t))$$

is a solution to the n-body problem where $q(t+T) = q(t)$.

Note that the entire solution is determined by the behavior of each mass over the time interval $[0, T/n]$. This is the *fundamental domain* for a choreography. The simplest choreography, discovered in 1772 by Lagrange [15], occurs when three equal masses are placed at the vertices of an equilateral triangle and given initial velocities so that the ensuing motion is rigid rotation about the centroid of the triangle. In this case, the curve swept out by the bodies is just a circle. Such a solution is an example of a *relative equilibrium*, a solution that is fixed in a rotating frame. The relative equilibria that generate circular choreographies are the regular n-gon solutions, that is, placing n equal masses at the vertices of a regular n-gon. These are the trivial choreographies. Amazingly, there exist an abundance of examples with non-trivial topology, the most famous of which is the figure-eight [9].

Theorem 1.2 *(Chenciner, Montgomery 2000) Fix a positive real number T. There exists a figure-eight shaped curve $\mathbf{q} : (\mathbb{R}/T\mathbb{Z}) \mapsto \mathbb{R}^2$ such that for all t*

1. $\mathbf{q}(t) + \mathbf{q}(t + T/3) + \mathbf{q}(t + 2T/3) = 0$
2. $\mathbf{q}(t + T/2) = -\bar{\mathbf{q}}(t),\ \mathbf{q}(-t + T/2) = \bar{\mathbf{q}}(t)$ *where $\bar{\mathbf{q}}$ denotes reflection of $\mathbf{q}$ about the x-axis.*
3. $(\mathbf{q}(t+2T/3), \mathbf{q}(t+T/3), \mathbf{q}(t))$ *is a zero angular momentum, periodic solution to the planar 3-body problem with equal masses.*

The two symmetries described in the second item arise from the construction of the orbit on the shape sphere and can be classified as a forward and time-reversing symmetry, respectively. Using the fact that the figure-eight is a choreography, these symmetries can be rewritten as

$$\mathbf{q}_1(t + \tfrac{T}{6}) = -\bar{\mathbf{q}}_2(t),\ \mathbf{q}_2(t + \tfrac{T}{6}) = -\bar{\mathbf{q}}_3(t),\ \mathbf{q}_3(t + \tfrac{T}{6}) = -\bar{\mathbf{q}}_1(t) \tag{1}$$

and

$$\mathbf{q}_1(-t + \tfrac{T}{6}) = \bar{\mathbf{q}}_1(t),\ \mathbf{q}_2(-t + \tfrac{T}{6}) = \bar{\mathbf{q}}_3(t),\ \mathbf{q}_3(-t + \tfrac{T}{6}) = \bar{\mathbf{q}}_2(t). \tag{2}$$

For later use, the initial conditions for the figure-eight orbit with $m_1 = m_2 = m_3 = 1$ are given by $\mathbf{q}_2(0) = -\mathbf{q}_1(0), \mathbf{q}_3(0) = (0,0)$ and $\dot{\mathbf{q}}_1(0) = \dot{\mathbf{q}}_2(0) = -\dot{\mathbf{q}}_3(0)/2$, where

$$\mathbf{q}_1(0) = \begin{bmatrix} 0.97000435669734 \\ -0.24308753153583 \end{bmatrix} \quad \text{and} \quad \dot{\mathbf{q}}_3(0) = \begin{bmatrix} -0.93240737144104 \\ -0.86473146092102 \end{bmatrix}.$$

The period of the orbit is $T = 6.32591398292621$ with the moment of inertia starting at $I(0) = 2$. These values were obtained from the data given by Simó in [20] using an appropriate scaling and angle of rotation.

1.2 Linear stability of periodic orbits

Throughout the paper, let

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

be the standard symplectic matrix, where I is the appropriately sized identity matrix. Suppose that $\zeta(t)$ is a T-periodic solution to the Hamiltonian system $\dot{z} = J\nabla H(z)$. Let $X(t)$ be the fundamental matrix solution to

$$\dot{\xi} = JD^2H(\zeta(t))\,\xi, \qquad \xi(0) = I. \tag{3}$$

$X(t)$ is symplectic and satisfies $X(t+T) = X(t)X(T)$ for all t. The matrix $X(T)$ is commonly referred to as the *monodromy matrix*, measuring the non-periodicity of solutions to the linearized equations. Its eigenvalues, the *characteristic multipliers*, determine the stability of the periodic solution. Being a symplectic matrix, the eigenvalues of $X(T)$ are symmetric with respect to the unit circle. Linear stability therefore requires that all of the multipliers lie on the unit circle.

It is important to note that the characteristic multipliers may be obtained by solving equation (3) with a different set of initial conditions. Suppose that Y_0 is an invertible matrix and that $Y(t)$ is the fundamental matrix solution to

$$\dot{\xi} = JD^2H(\zeta(t))\,\xi, \qquad \xi(0) = Y_0. \tag{4}$$

By definition of $X(t)$, we have that $Y(t) = X(t)Y_0$ and consequently, $X(T) = Y(T)Y_0^{-1}$. It follows that the matrix $Y_0^{-1}Y(T)$ is similar to the monodromy matrix so that the eigenvalues of $Y_0^{-1}Y(T)$ are identical to the characteristic multipliers. This will be useful for making a good change of coordinates for the figure-eight orbit.

Every integral in the n-body problem yields a multiplier of $+1$. For a periodic orbit in the planar problem there are always eight $+1$ multipliers, four arising from the center of mass and total linear momentum integrals and four others due to the $SO(2)$ symmetry, the angular momentum integral, the Hamiltonian and the periodic orbit itself. Moreover, because of drift, the Jordan blocks may contain off diagonal terms. In the classical sense, every periodic solution of the n-body problem is unstable due to this drift. In light of these degeneracies, it is natural to define the linear stability of a periodic solution by examining stability on the reduced quotient space.

Definition 1.3 *A periodic solution of the planar n-body problem has 8 trivial characteristic multipliers of $+1$. The solution is* **spectrally stable** *if the remaining multipliers lie on the unit circle and* **linearly stable** *if in addition, the monodromy matrix $X(T)$ restricted to the reduced space is diagonalizable.*

2 Stability reductions using symmetry

The monodromy matrix for a periodic solution with special types of symmetry can be factored using some linear algebra and standard techniques in differential equations. For example, it is only necessary to study the linearized equations for a choreography over its fundamental domain $[0, T/n]$. Our goal is to make use of the forward and time-reversing symmetries of the figure-eight orbit in order to reduce the stability calculations down to the first twelfth of the orbit. We begin by stating some useful reductions applicable to a wide range of symmetric periodic orbits commonly found in Hamiltonian systems.

Lemma 2.1 *Suppose that $\gamma(t)$ is a a symmetric T-periodic solution of a Hamiltonian system with Hamiltonian H and symmetry matrix S such that*

1. *For some $N \in \mathbb{N}$, $\gamma(t + T/N) = S\gamma(t) \;\; \forall t$*
2. *$H(Sx) = H(x)$*
3. *$SJ = JS$*
4. *S is orthogonal.*

Then the fundamental matrix solution $X(t)$ to the linearization problem $\dot{\xi} = JD^2H(\gamma(t))\,\xi,\; \xi(0) = I$ satisfies

$$X(t + \tfrac{T}{N}) = SX(t)S^{\mathrm{T}}X(\tfrac{T}{N}).$$

Proof: Let $A = X(T/N)$, $Y(t) = X(t + T/N)$ and $Z(t) = SX(t)S^{\mathrm{T}}X(T/N)$. Consider the differential equation

$$\dot{\xi} = SJD^2H(\gamma(t))S^{\mathrm{T}}\,\xi, \quad \xi(0) = A. \tag{5}$$

We compute $\dot{Y} = \dot{X}(t{+}T/N) = JD^2H(\gamma(t{+}T/N))X(t{+}T/N) = SJD^2H(\gamma(t))S^{\mathrm{T}}Y(t)$ using Properties $1, 2$ and 3 of S. On the other hand, $\dot{Z} = SJD^2H(\gamma(t))X(t)S^{\mathrm{T}}A = SJD^2H(\gamma(t))S^{\mathrm{T}}Z(t)$ using the fact that S is orthogonal. Thus, both $Y(t)$ and $Z(t)$ satisfy equation (5) with the same initial condition $Y(0) = Z(0) = A$. By the uniqueness theorem for differential equations, $Y(t) \equiv Z(t)$ as required. □

Corollary 2.2 *Given the hypotheses of Lemma 2.1, the fundamental matrix solution $X(t)$ satisfies*

$$X(\tfrac{kT}{N}) = S^k(S^{\mathrm{T}}X(\tfrac{T}{N}))^k.$$

for any $k \in \mathbb{N}$.

Proof: The formula for $X(kT/N)$ follows from Lemma 2.1 by induction on k. □

Remarks:

1. Here, the symmetry S is a constant square matrix of appropriate size. In the planar n-body problem, S is of dimension $4n \times 4n$. S is typically block diagonal with two equivalent blocks, one for the position variables and one for the momenta, that is,

$$S = \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}$$

   where R is orthogonal. A matrix of this form is both orthogonal and commutes with J.

2. A matrix satisfying properties (3) and (4) of Lemma 2.1 is always symplectic since $S^{\mathrm{T}}JS = S^{\mathrm{T}}SJ = J$. Such a matrix is called *unitary* in the language of symplectic geometry.

3. If $S^N = I$, then the monodromy matrix for ζ factors as $(S^{\mathrm{T}}X(T/N))^N$. This is particularly useful for choreographies (see Corollary 2.3).

4. In the case $n = 1$, when S is a symmetry over the entire periodic orbit, Lemma 2.1 shows that S commutes with $X(t) \quad \forall t$.

5. If $Y(t)$ is the fundamental matrix solution to equation (4), where $\xi(0) = Y_0$, then a similar argument shows that $Y(t + T/N) = SY(t)Y_0^{-1}S^{\mathrm{T}}Y(T/N)$ and consequently

$$Y(\tfrac{kT}{N}) = S^k Y_0 \left(Y_0^{-1}S^{\mathrm{T}}Y(\tfrac{T}{N})\right)^k. \tag{6}$$

Lemma 2.1 and Corollary 2.2 are easily applied to any equal-mass choreography. Let σ be the permutation matrix such that $\sigma(\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_n)^{\mathrm{T}} = (\mathbf{q}_n, \mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_{n-1})^{\mathrm{T}}$ and consider the block diagonal matrix P given by

$$P = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}.$$

Corollary 2.3 *Suppose that $\gamma(t)$ is an equal-mass choreography and let $X(t)$ be the fundamental matrix solution to the linearized equations about $\gamma(t)$. Then the monodromy matrix for γ factors as*

$$X(T) \;=\; (P^{\mathrm{T}}X(\tfrac{T}{n}))^n.$$

Consequently, the linear stability of γ is determined by the eigenvalues of $P^{\mathrm{T}}X(T/n)$.

Proof: The matrix P is orthogonal, symplectic and commutes with J. Since the masses are equal, it is also an invariant for the Hamiltonian H of the n-body problem. By definition of a choreography, we have $\gamma(t + T/n) = \sigma\,\gamma(t) \quad \forall t$. The same equation is true if γ is replaced by $\dot\gamma$. Thus the hypotheses of Lemma 2.1 are satisfied with $S = P$ and $N = n$. Using the fact that $P^n = I$, setting $k = n$ in Corollary 2.2 yields the result. □

Lemma 2.4 *Suppose that $\gamma(t)$ is a T-periodic solution of a Hamiltonian system with Hamiltonian H and time-reversing symmetry S such that*

1. *For some $N \in \mathbb{N}$, $\gamma(-t + T/N) = S\gamma(t)\ \forall t$*
2. *$H(Sx) = H(x)$*
3. *$SJ = -JS$*
4. *S is orthogonal.*

Then the fundamental matrix solution $X(t)$ to the linearization problem $\dot\xi = JD^2H(\gamma(t))\,\xi, \quad \xi(0) = I$ satisfies

$$X(-t + \tfrac{T}{N}) = SX(t)S^{\mathrm{T}}X(\tfrac{T}{N}). \tag{7}$$

Proof: The proof is similar to that of Lemma 2.1. Let $A = X(T/N)$, $Y(t) = X(-t + T/N)$ and $Z(t) = SX(t)S^{\mathrm{T}}X(T/N)$. Consider the differential equation

$$\dot{\xi} = SJD^2H(\gamma(t))S^{\mathrm{T}}\,\xi, \quad \xi(0) = A. \tag{8}$$

We compute $\dot{Y} = -\dot{X}(-t+T/N) = -JD^2H(\gamma(-t+T/N))X(-t+T/N) = SJD^2H(\gamma(t))S^{\mathrm{T}}Y(t)$ using Properties 1, 2 and 3 of S. On the other hand, $\dot{Z} = SJD^2H(\gamma(t))X(t)S^{\mathrm{T}}A = SJD^2H(\gamma(t))S^{\mathrm{T}}Z(t)$ using the fact that S is orthogonal. Thus, both $Y(t)$ and $Z(t)$ satisfy equation (8) with the same initial conditions. By the uniqueness theorem for differential equations, $Y(t) \equiv Z(t)$ as required. □

Corollary 2.5 *Given the hypotheses of Lemma 2.4,*

$$X(\tfrac{T}{N}) = SB^{-1}S^{\mathrm{T}}B, \quad \textit{where } B = X(\tfrac{T}{2N}).$$

Proof: Evaluating formula (7) at $t = T/2N$ gives $B = SBS^{\mathrm{T}}X(T/N)$. Solving this equation for $X(T/N)$ gives the desired formula. □

Remarks:

1. In the case of time-reversal symmetry, S is typically block diagonal with two blocks of opposite sign, one for the position variables and one for the momenta, that is,

$$S = \begin{bmatrix} R & 0 \\ 0 & -R \end{bmatrix}$$

   where R is orthogonal. A matrix of this form is orthogonal and anti-commutes with J.

2. A matrix satisfying properties (3) and (4) of Lemma 2.4 is symplectic with a multiplier of -1 since $S^{\mathrm{T}}JS = -S^{\mathrm{T}}SJ = -J$.

3. In the case $n = 1$, Lemma 2.4 yields $SX(t) = X(-t)S \quad \forall t$.

4. If $Y(t)$ is the fundamental matrix solution to equation (4), where $\xi(0) = Y_0$, then a similar argument shows that $Y(-t + T/N) = SY(t)Y_0^{-1}S^{\mathrm{T}}Y(T/N)$ and consequently

$$Y(\tfrac{T}{N}) = SY_0B^{-1}S^{\mathrm{T}}B, \quad \text{where } B = Y(\tfrac{T}{2N}). \tag{9}$$

3 Changing coordinates

To eliminate the trivial $+1$ multipliers of the figure-eight orbit, we use shape sphere coordinates, following the construction of the figure-eight by Chenciner and Montgomery in [9]. This sphere is the space of oriented similarity classes of triangles, that is, the space of triangles quotiented out by scaling, translation and rotation. After projection onto the sphere, twelve copies of the fundamental piece of the figure-eight are glued together smoothly to form the full orbit. The symmetries of the orbit are thus naturally described in these coordinates.

3.1 Shape sphere coordinates

We begin with scaled Jacobi coordinates. Set $m_i = 1\ \forall i$ and denote $\mathbf{p}_i = \dot{\mathbf{q}}_i$ as the momentum coordinates. Then let

$$\begin{aligned} \mathbf{u}_1 &= \tfrac{1}{\sqrt{2}}(\mathbf{q}_3 - \mathbf{q}_2) & \mathbf{v}_1 &= \tfrac{1}{\sqrt{2}}(\mathbf{p}_3 - \mathbf{p}_2) \\ \mathbf{u}_2 &= \sqrt{\tfrac{2}{3}}(\mathbf{q}_1 - \tfrac{1}{2}(\mathbf{q}_2 + \mathbf{q}_3)) & \mathbf{v}_2 &= \sqrt{\tfrac{2}{3}}(\mathbf{p}_1 - \tfrac{1}{2}(\mathbf{p}_2 + \mathbf{p}_3)) \\ \mathbf{u}_3 &= \tfrac{1}{3}(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3) & \mathbf{v}_3 &= \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3. \end{aligned}$$

The new Hamiltonian $H(\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2) = (||\mathbf{v}_1||^2 + ||\mathbf{v}_2||^2)/2 - U(\mathbf{u}_1, \mathbf{u}_2)$ is independent of $\mathbf{u}_3$ and $\mathbf{v}_3$, the center of mass and total linear momentum, respectively. In these coordinates the inertia becomes $I = ||\mathbf{u}_1||^2 + ||\mathbf{u}_2||^2$. We also have $r_{12} = ||\sqrt{(3/2)}\,\mathbf{u}_2 + \sqrt{(1/2)}\,\mathbf{u}_1||$, $r_{13} = ||\sqrt{(3/2)}\,\mathbf{u}_2 - \sqrt{(1/2)}\,\mathbf{u}_1||$ and $r_{23} = \sqrt{2}\,||\mathbf{u}_1||$. This reduces the dimension by 4 from 12 to 8.

To obtain coordinates on the shape sphere, we use the Hopf map to generate a point transformation $w = \phi(u)$. For notational purposes, $\mathbf{u} \cdot \mathbf{v}$ refers to the usual dot product in $\mathbb{R}^2$. We treat $\mathbf{u} \times \mathbf{v}$ as a scalar obtained by taking the nonzero component of the cross product of two vectors in $\mathbb{R}^2$. Coordinates on the full phase space are derived by extending ϕ to a symplectic transformation in the natural way, $\mathbf{z} = (\partial\phi/\partial w)^{-\mathrm{T}}\mathbf{v}$ where $-\mathrm{T}$ denotes the inverse of the transpose. Our new variables are the eight one-dimensional variables w_i, z_i given by

$$\begin{aligned} w_1 &= ||\mathbf{u}_1||^2 - ||\mathbf{u}_2||^2 & z_1 &= \tfrac{1}{2I}(\mathbf{u}_1 \cdot \mathbf{v}_1 - \mathbf{u}_2 \cdot \mathbf{v}_2) \\ w_2 &= 2(\mathbf{u}_1 \cdot \mathbf{u}_2) & z_2 &= \tfrac{1}{2I}(\mu\, \mathbf{u}_1 \cdot \mathbf{v}_1 - \nu\, \mathbf{u}_2 \times \mathbf{v}_2 + \mathbf{u}_1 \cdot \mathbf{v}_2) \\ w_3 &= 2(\mathbf{u}_1 \times \mathbf{u}_2) & z_3 &= \tfrac{1}{2I}(\nu\, \mathbf{u}_1 \cdot \mathbf{v}_1 + \mu\, \mathbf{u}_2 \times \mathbf{v}_2 + \mathbf{u}_1 \times \mathbf{v}_2) \\ w_4 &= \arg(\mathbf{u}_1) & z_4 &= \mathbf{u}_1 \times \mathbf{v}_1 + \mathbf{u}_2 \times \mathbf{v}_2 \end{aligned}$$

where $\mu = \mathbf{u}_1 \cdot \mathbf{u}_2/||\mathbf{u}_1||^2$ and $\nu = \mathbf{u}_1 \times \mathbf{u}_2/||\mathbf{u}_1||^2$.

The new Hamiltonian is independent of w_4 so that $z_4 = c$ is a constant of motion. One can check that $z_4 = \sum_{i=1}^{3} \mathbf{q}_i \times \mathbf{p}_i$ is the usual angular momentum. Letting $w = (w_1, w_2, w_3)$ and $z = (z_1, z_2, z_3)$, the new Hamiltonian is $H(w, z) = 2K(z)I(w) - U(w) + c(c + 2w_3z_2 - 2w_2z_3)/(I(w) + w_1)$ where $K(z) = z_1^2 + z_2^2 + z_3^2$, $I(w) = (w_1^2 + w_2^2 + w_3^2)^{1/2}$ and

$$U(w) = \frac{1}{\sqrt{I + w_1}} + \frac{1}{\sqrt{I - \frac{1}{2}w_1 + \frac{\sqrt{3}}{2}w_2}} + \frac{1}{\sqrt{I - \frac{1}{2}w_1 - \frac{\sqrt{3}}{2}w_2}}.$$

We have reduced the problem to three degrees of freedom, the usual dimension of the planar three-body problem. From these coordinates it is not difficult to understand the shape sphere $I = 1$. The cross product $\mathbf{u}_1 \times \mathbf{u}_2$ vanishes if and only if the three bodies are collinear. Therefore, the collinear configurations correspond to the equator $w_3 = 0$. The two oppositely oriented equilateral triangles are located at the North and South poles $(0, 0, \pm 1)$. There are three meridians M_i, each passing through the North and South poles, that correspond to an isosceles triangle with mass m_i at the apex. For later use, the meridian M_1 is the great circle $w_2 = 0$.

3.2 Instability of the Lagrange solution

In these new coordinates, the Lagrange relative equilibrium solution $\gamma(t)$ is rather simple. Given a fixed $\alpha > 0$ representing the size of the equilateral triangle, choose the angular momentum $c > 0$ such that $c^2 = 3\sqrt{\alpha}$. The periodic solution reduces to the fixed point $\gamma = (0, 0, \alpha, 0, -c/(2\alpha), 0)$ and it is straight-forward to calculate the characteristic multipliers. The actual motion (rigid rotation) is hidden in the cyclic coordinate $w_4 = \omega t$ where $\omega = c/\alpha$. The linearized equation $\dot{\xi} = JD^2H(\gamma)\xi$ is no longer time dependent. A short computation gives

$$JD^2H(\gamma) = \begin{bmatrix} 0 & 0 & 0 & 4\alpha & 0 & 0 \\ -2\omega & 0 & -2\omega & 0 & 4\alpha & 0 \\ 0 & -2\omega & 0 & 0 & 0 & 4\alpha \\ -\frac{5\omega^2}{8\alpha} & 0 & -\frac{\omega^2}{\alpha} & 0 & 2\omega & 0 \\ 0 & -\frac{5\omega^2}{8\alpha} & 0 & 0 & 0 & 2\omega \\ -\frac{\omega^2}{\alpha} & 0 & -\frac{5\omega^2}{4\alpha} & 0 & 2\omega & 0 \end{bmatrix}.$$

The abundance of zeroes in this matrix is a result of the symmetry of the orbit. Using the calculations of the previous section, the matrix $S = \text{diag}\,\{1, -1, 1, -1, 1, -1\}$ anti-commutes with $JD^2H(\gamma)$. Any such matrix A must contain zeroes on every "even" entry, that is, $a_{ij} = 0$ whenever $i + j = 0 \bmod 2$.

By writing the matrix $JD^2H(\gamma)$ as a block diagonal matrix, a short calculation gives the characteristic polynomial of $JD^2H(\gamma)$ as $p(\lambda) = (\lambda^2 + \omega^2)(\lambda^4 + \omega^2\lambda^2 + \frac{9}{4}\omega^4)$. This agrees with previous stability analyses of the equilateral triangle relative equilibrium (see [16, 19] for example). If λ is a root of p, then $e^{2\pi\lambda/\omega}$ is a characteristic multiplier for the Lagrange solution. The first factor of p yields the eigenvalues $\pm\omega\, i$ and the two remaining trivial $+1$ multipliers. The second factor of p has four roots of the form

$$\pm\sqrt{-\tfrac{1}{2} \pm \sqrt{2}\, i}\;\; \omega \;=\; \pm\left(\frac{1}{\sqrt{2}} \pm i\right)\,\omega$$

yielding two equal pairs of characteristic multipliers $e^{\pm\sqrt{2}\,\pi}$. Note that $e^{\sqrt{2}\,\pi} \approx 85.02$ so these multipliers are well off the unit circle.

Theorem 3.1 *The nontrivial characteristic multipliers of the Lagrange equilateral triangle solution with equal masses are* $e^{\pm\sqrt{2}\,\pi}, e^{\pm\sqrt{2}\,\pi}$.

3.3 Stability reductions for the figure-eight

In stark contrast to the circular choreography for three bodies, the multipliers for the figure-eight orbit lie on the unit circle. For the remainder of this work, let $\gamma(t)$ be the figure-eight periodic solution and let $X(t)$ be the fundamental matrix solution for the linearized equations. Setting the angular momentum $c = 0$, the Hamiltonian reduces nicely to $H = 2KI - U$. We will use the two special symmetries S_f and S_r, the forward and time-reversing symmetries of the figure-eight, respectively, and Corollaries 2.2 and 2.5 to factor the monodromy matrix.

The symmetry S_f is obtained by rotating the first sixth of the figure-eight 120° around the shape sphere followed by a reflection about the equator. This can be derived by writing equation (1) in our new coordinates. Note that I is preserved under this transformation. Using the equations of motion, we have $z_i = \dot{w}_i/4I$, so that the same transformation is applied to the momenta. This gives

$$\gamma(t + \tfrac{T}{6}) \;=\; S_f\,\gamma(t) \quad \forall t \tag{10}$$

where

$$S_f = \begin{bmatrix} R_f & 0 \\ 0 & R_f \end{bmatrix} \qquad \text{and} \qquad R_f = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

The matrix S_f is orthogonal, commutes with J and satisfies $S_f^6 = I$. Moreover, one can check that $H(R_f w, R_f z) = H(w, z)$. Using Corollary 2.2, the monodromy matrix factors as

$$X(T) = \left(S_f^{\mathrm{T}}\, X(\tfrac{T}{6})\right)^6.$$

For completeness, we note that the permutation symmetry arising from γ as a choreography is contained in S_f. The identity $\gamma(t+T/3) = S_f^2\,\gamma(t)$ leads to a cube root rather than a sixth root of the monodromy matrix. The matrix R_f^2 is a 240° rotation of the shape sphere, which is equivalent to the permutation σ in our new coordinates.

The time-reversal symmetry S_r is obtained by a reflection about the meridian M_1 given by $w_2 = 0$. This symmetry is particularly simple on the shape sphere and can be checked by writing equation (2) in our new coordinates. The inertia I is also preserved under this transformation. Since this is a time-reversing symmetry, the negative of the reflection is applied to the momenta. Specifically, we have

$$\gamma(-t + \tfrac{T}{6}) = S_r\,\gamma(t) \quad \forall t$$

where

$$S_r = \begin{bmatrix} R_r & 0 \\ 0 & -R_r \end{bmatrix} \qquad \text{and} \qquad R_r = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The matrix S_r is symmetric, orthogonal, anti-commutes with J and satisfies $S_r^2 = I$. Moreover, it is easy to see that $H(R_r w, -R_r z) = H(w, z)$. Using Corollary 2.5, we obtain

$$X(\tfrac{T}{6}) = S_r A^{-1} S_r A, \qquad A = X(\tfrac{T}{12})$$

and thus the monodromy matrix factors into $X(T) = \left(S_f^{\mathrm{T}} S_r A^{-1} S_r A\right)^6$. Letting $Q = S_f^{\mathrm{T}} S_r$, we have proven:

Theorem 3.2 *The monodromy matrix for the figure-eight orbit is* $(QA^{-1}S_rA)^6$ *where* $A = X(T/12), S_r =$ *diag* $\{1, -1, 1, -1, 1, -1\}$,

$$Q = \begin{bmatrix} R & 0 \\ 0 & -R \end{bmatrix} \quad \textit{and} \quad R = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

4 Linear stability analysis

We will call a symplectic matrix *stable* if its eigenvalues lie on the unit circle. We have factored the monodromy matrix for the eight, showing precisely how the dynamics of the first twelfth of the orbit determine its linear stability. This factorization has some special properties.

Denote $B = A^{-1} S_r A$. Both Q and B are symplectic with multiplier -1 and consequently the matrix QB is symplectic with multiplier $+1$. The eigenvalues of both Q and B are $\pm 1, \pm 1, \pm 1$. The matrix Q is orthogonal and symmetric and both matrices are involutions, $Q^2 = B^2 = I$. However, very little is known about the eigenvalues of the product of two matrices. In general, stability is not preserved under multiplication. Even in the 2×2 case, there is no direct correlation between the stability of $QB = S_r A^{-1} S_r A$ and the stability of A.

On the other hand, it does appear that the time-reversal symmetry and hence the inclusion of A^{-1} in the factorization give the figure-eight a *chance* at being stable. Support for this notion comes from a family of orbits in the equal mass 3-body problem discovered by Hénon which are linearly stable and exhibit a time-reversal symmetry similar to the figure-eight [13]. Another family of symmetric orbits, not necessarily with equal masses, has been recently announced by Chen, et al. which also contains linearly stable members [6]. We conjecture that the only stable equal mass periodic solutions are those that exhibit some type of time-reversing symmetry.

4.1 A good basis

There is an important reduction that can be achieved by choosing a good basis over which to solve the linearized equations. Let $Y(t)$ be the fundamental matrix solution to the linearized equations about the figure-eight with arbitrary initial conditions Y_0. As stated in Section 1.2, the characteristic multipliers are unchanged by a shift in initial conditions and are given by the eigenvalues of $Y_0^{-1} Y(T)$, which is essentially the new monodromy matrix. Equation (6) gives us $Y_0^{-1} Y(T) = W^6$ where

$$W \;=\; Y_0^{-1} S_f^{\mathrm{T}} Y(\tfrac{T}{6}). \tag{11}$$

Then, equation (9) yields $W = Y_0^{-1} S_f^{\mathrm{T}} S_r Y_0 C^{-1} S_r C, \;\; C = Y(T/12)$. Thus the stability problem reduces to showing that the eigenvalues of

$$W = Y_0^{-1} Q Y_0 \, C^{-1} S_r C$$

are on the unit circle.

Lemma 4.1 *Suppose that the symplectic matrix W satisfies the property that*

$$\frac{1}{2}\left(W + W^{-1}\right) \;=\; \begin{bmatrix} K^{\mathrm{T}} & 0 \\ 0 & K \end{bmatrix}. \tag{12}$$

Then W is stable if and only if all of the eigenvalues of K are real and between -1 and 1.

Proof: Suppose that $\mathbf{v}$ is an eigenvector for W with eigenvalue λ. Then $(1/2)(W + W^{-1})\mathbf{v} = (1/2)(\lambda + 1/\lambda)\mathbf{v}$ from which it follows that $(1/2)(\lambda + 1/\lambda)$ is an eigenvalue of K. The map $f : \mathbb{C} \mapsto \mathbb{C}$ given by $f(\lambda) = (1/2)(\lambda + 1/\lambda)$ takes the unit circle onto the real interval $[-1, 1]$ while mapping the exterior of the unit disk homeomorphically onto $\mathbb{C} - [-1, 1]$.

First, suppose that all of the eigenvalues of K lie in the real interval $[-1, 1]$. If W were not stable, then W would have an eigenvalue λ with modulus greater than one. But this implies that K has an eigenvalue $f(\lambda)$ outside $[-1, 1]$, a contradiction. Therefore, W must be stable.

Conversely, suppose that W is stable. This means that the eigenvalues of $(1/2)(W + W^{-1})$ lie in the real interval $[-1, 1]$. Let μ be an eigenvalue of K. Then $K\mathbf{u} = \mu\mathbf{u}$ implies that

$$\frac{1}{2}\left(W + W^{-1}\right) \begin{bmatrix} 0 \\ \mathbf{u} \end{bmatrix} \;=\; \mu \begin{bmatrix} 0 \\ \mathbf{u} \end{bmatrix}.$$

Therefore, μ is an eigenvalue of $(1/2)(W + W^{-1})$ and must be real and between -1 and 1. $\square$

We claim it is possible to put W into the special form of Lemma 4.1 by choosing Y_0 appropriately. Specifically, we take Y_0 to be orthogonal, symplectic, containing the eigenvectors of Q and such that the fourth column of Y_0 is $\dot{\gamma}(0)/||\dot{\gamma}(0)||$. We also diagonalize Q so that $Y_0^{-1} Q Y_0 = \Lambda = \text{diag}\,\{1, 1, 1, -1, -1, -1\}$.

In wz-coordinates, the figure-eight begins at $w(0) = (-1, \sqrt{3}, 0)$ and $z(0) = (\alpha, \alpha/\sqrt{3}, \beta)$ where $\alpha = (3/8)\,\mathbf{q}_1(0) \cdot \dot{\mathbf{q}}_3(0)$ and $\beta = (-\sqrt{3}/4)\,\mathbf{q}_1(0) \times \dot{\mathbf{q}}_3(0)$. Using the equations of motion, we have $\dot{\gamma}(0) = [\sqrt{3}\,a, a, b, c, -\sqrt{3}\,c, 0]^{\mathrm{T}}$ where

$$a = \frac{8\alpha}{\sqrt{3}}, \quad b = 8\beta \quad \text{and} \quad c = \frac{3||\dot{\mathbf{q}}_3(0)||^2 + 5}{16}.$$

Next, define vectors $\mathbf{u}_2 = [\sqrt{3}\,b, b, -4a, 0, 0, 0]^{\mathrm{T}}$ and $\mathbf{u}_3 = [\sqrt{3}\,a, a, b, d, -\sqrt{3}\,d, 0]^{\mathrm{T}}$ where $d = -(4a^2 + b^2)/4c$. Setting $\mathbf{v}_1 = \dot{\gamma}(0)/||\dot{\gamma}(0)||$, $\mathbf{v}_2 = \mathbf{u}_2/||\mathbf{u}_2||$ and $\mathbf{v}_3 = \mathbf{u}_3/||\mathbf{u}_3||$ yields an orthonormal set of vectors in the eigenspace of -1 for Q. The standard symplectic matrix J takes the eigenspace of -1 for Q into the $+1$ eigenspace for Q. We then take the 6 columns of Y_0 to be $Y_0 = [J\mathbf{v}_1, J\mathbf{v}_2, J\mathbf{v}_3, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ yielding an orthogonal, symplectic matrix which diagonalizes Q into $\Lambda = \text{diag}\,\{1, 1, 1, -1, -1, -1\}$. The matrix Y_0 is of the form

$$Y_0 = \begin{bmatrix} P_1 & P_2 \\ -P_2 & P_1 \end{bmatrix}$$

where $P_1^{\mathrm{T}} P_1 + P_2^{\mathrm{T}} P_2 = I$ and $P_1^{\mathrm{T}} P_2 = 0$. A matrix of this form is both symplectic and orthogonal.

With this choice of Y_0, the stability matrix W has been reduced to the special form $\Lambda C^{-1} S_r C$ where C is symplectic and $C = Y(T/12)$. Letting $D = C^{-1} S_r C$, a short computation using the formula for the inverse of a symplectic matrix shows that D has the form

$$D = \begin{bmatrix} K^{\mathrm{T}} & L_1 \\ -L_2 & -K \end{bmatrix}$$

where L_1 and L_2 are 3×3 symmetric matrices. Note that $\Lambda^2 = D^2 = I$ so that $D\Lambda$ is the inverse of $\Lambda D = W$. It follows that W has the special property given by equation (12). By Lemma 4.1, the figure-eight is spectrally stable if and only if the eigenvalues of K are real and between -1 and 1.

By construction, K has the form

$$\begin{bmatrix} 1 & * & * \\ 0 & \mathbf{c}_2 \cdot S_r J \mathbf{c}_5 & \mathbf{c}_2 \cdot S_r J \mathbf{c}_6 \\ 0 & \mathbf{c}_3 \cdot S_r J \mathbf{c}_5 & \mathbf{c}_3 \cdot S_r J \mathbf{c}_6 \end{bmatrix}$$

where $\mathbf{c}_i$ is the ith column of $C = Y(T/12)$ and "$\cdot$" is the standard dot product. To see this note that differentiating $\dot{\gamma} = J\nabla H(\gamma)$ yields $\ddot{\gamma} = JD^2 H(\gamma(t))\dot{\gamma}$. This implies that $\dot{\gamma}(t)$ satisfies the associated linear system

$$\dot{\xi} = JD^2 H(\gamma(t))\,\xi, \quad \xi(0) = \dot{\gamma}(0).$$

Since $Y(t)$ satisfies the linearized differential equations as well, we have

$$Y(t) Y_0^{-1} \dot{\gamma}(0) = \dot{\gamma}(t) \;\; \forall t. \tag{13}$$

Let $\mathbf{v} = Y_0^{-1} \dot{\gamma}(0)$. Using the symmetry (10) and equations (11) and (13), we see that

$$W\mathbf{v} = Y_0^{\mathrm{T}} S_f^{\mathrm{T}} Y(\tfrac{T}{6})\,\mathbf{v} = Y_0^{\mathrm{T}} S_f^{\mathrm{T}} \dot{\gamma}(\tfrac{T}{6}) = Y_0^{\mathrm{T}} S_f^{\mathrm{T}} S_f \dot{\gamma}(0) = \mathbf{v}.$$

Thus, v is an eigenvector of W with eigenvalue $+1$. This $+1$ multiplier arises from the perturbation in the direction of the periodic orbit. The above calculation shows that v is an eigenvector of the monodromy matrix as well.

Finally, since Y_0 is orthogonal, the vector

$$\mathbf{v} = Y_0^{-1}\dot{\gamma}(0) = Y_0^{\mathrm{T}}\dot{\gamma}(0) = ||\dot{\gamma}(0)||\, \mathbf{e}_4$$

is a multiple of the fourth column of the 6×6 identity matrix. Hence $W\mathbf{e}_4 = \mathbf{e}_4$ and the first column of K is $[1, 0, 0]^{\mathrm{T}}$. The rest of the form of K comes from the formula for the inverse of a symplectic matrix and $D = C^{-1}S_r C$. We have reduced the stability calculations for the figure-eight orbit down to a 2×2 matrix whose entries only depend on the first twelfth of the orbit.

Theorem 4.2 *Let $\mathbf{c}_i$ be the ith column of $C = Y(T/12)$, where $Y(t)$ is the fundamental matrix solution to the linearized equations for the figure-eight with special initial conditions Y_0. Then the figure-eight is spectrally stable if and only if the eigenvalues of the 2×2 matrix*

$$\begin{bmatrix} \mathbf{c}_2 \cdot S_r J\mathbf{c}_5 & \mathbf{c}_2 \cdot S_r J\mathbf{c}_6 \\ \mathbf{c}_3 \cdot S_r J\mathbf{c}_5 & \mathbf{c}_3 \cdot S_r J\mathbf{c}_6 \end{bmatrix} \tag{14}$$

lie in the real interval $[-1, 1]$.

4.2 Numerical calculations

Using MATLAB version 6.5.1 and its built-in differential equation solver `ode45`, we compute the matrix $C = Y(T/12)$ with an absolute error tolerance of 1×10^{-14}. From this we compute K easily and calculate its eigenvalues to be

$$\lambda_1 = 1, \quad \lambda_2 = 0.20986512354505, \quad \lambda_3 = -0.50761901821201.$$

Since these values are distinct, Theorem 4.2 then shows that the figure-eight is linearly stable. Returning to the full monodromy matrix, the values for λ_2 and λ_3 lead to the same non-trivial eigenvalues computed by Simó in [20]. Note that $\lambda_{2,3} \neq \pm 1/2$ and consequently, the characteristic multipliers are not $+1$, providing a nondegeneracy argument for the figure-eight orbit.

To obtain rigorous numerical estimates using as few steps as possible, a MATLAB routine was written using a Runge-Kutta-Fehlberg method with local truncation error of order four (see [3] pp. 286). The goal is to do as few iterations as possible but still be able to conclude stability within rigorous error estimates.

The eigenvalues of K are determined by the trace τ and determinant Δ of (14). Necessary and sufficient conditions for the eigenvalues to be real and lie in the interval $[-1, 1]$ are given by the four conditions $\Delta \leq \tau^2/4, \Delta \geq -\tau - 1, \Delta \geq \tau - 1$ and $-2 \leq \tau \leq 2$. This determines a compact region of the $\tau\Delta$-plane shown in Figure 2. Note the similarity of this region with the stability region in the ab-plane of a reciprocal quartic polynomial $p(z) = z^4 + az^3 + bz^2 + az + 1$, shown in Figure 1 of [19] for example.

Define the three quantities s_1, s_2, s_3 by

$$\begin{aligned} s_1 &= \Delta + \tau + 1 \\ s_2 &= \Delta - \tau + 1 \\ s_3 &= \tfrac{1}{4}\tau^2 - \Delta. \end{aligned}$$

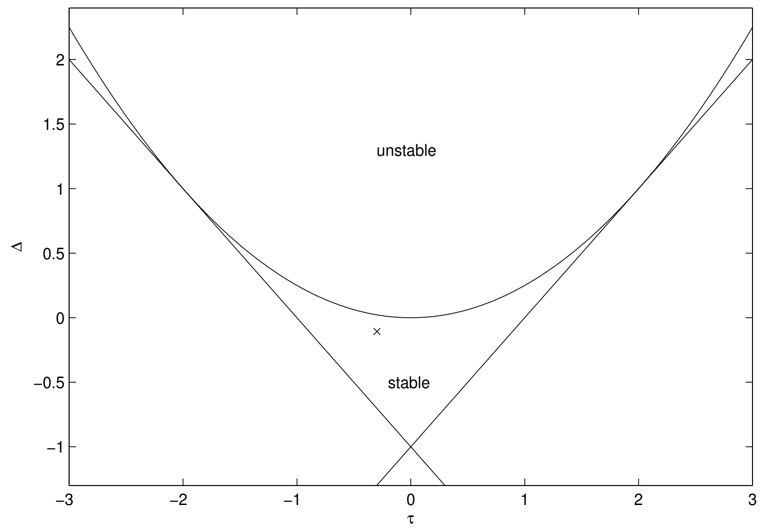

Figure 2: The stability region for a 2×2 matrix with trace τ and determinant Δ to have real eigenvalues in the interval $[-1, 1]$. An X marks the location of the eigenvalues of (14) for the figure-eight orbit.

It follows that the eigenvalues of (14) are real and lie in the interval $[-1, 1]$ if and only if $s_i \geq 0 \; \forall i$ and $-2 \leq \tau \leq 2$. We find rigorous estimates for these four crucial quantities as follows.

Let c_{ij} be the entry on the ith row, jth column of C. Using the fact that C is symplectic, matrix (14) reduces to

$$\begin{bmatrix} 2(c_{12}c_{45} + c_{32}c_{65} - c_{52}c_{25}) - 1 & 2(c_{12}c_{46} + c_{32}c_{66} - c_{52}c_{26}) \\ 2(c_{13}c_{45} + c_{33}c_{65} - c_{53}c_{25}) & 2(c_{13}c_{46} + c_{33}c_{66} - c_{53}c_{26}) - 1 \end{bmatrix} \tag{15}$$

Of the 36 entries in $C = Y(T/12)$, only 12 are needed to calculate stability.

Assuming that the local truncation error ϵ used in the Runge-Kutta-Fehlberg method is a bound for the error on the entries of C, we can use the formulas above to obtain rigorous error bounds for τ, Δ and s_1, s_2, s_3. For example, the estimated value for τ will be within E_τ of the actual value, where

$$E_\tau = 2\epsilon \left(\sum |c_{**}| + 6\epsilon \right)$$

and the sum is over all 12 indices included on the diagonal of matrix (15). A similar, but slightly more involved expression exists to find the error E_Δ for the determinant Δ. It then follows that the stability quantities s_1 and s_2 will be accurate to within $E_{s_{12}} = E_\tau + E_\Delta$ while the quantity s_3 is accurate to within $E_{s_3} = E_\tau(2|\tau| + E_\tau)/4 + E_\Delta$. Using these estimates, a local truncation error of $\epsilon = 0.004$ is just accurate enough to yield nonnegative values for the s_i as well as $\tau \in [-2, 2]$ within the given error tolerances. In this case, a maximum of only 4 steps of the Runge-Kutta-Fehlberg algorithm were used. See Table 1 for the numerical results of other tolerances. It is clear that our reductions from symmetry and choice of coordinates significantly decreases the amount of numerical calculations needed to conclude linear stability of the figure-eight orbit.

ϵ	m	τ	s_1	s_2	s_3	E_τ	$E_{s_{12}}$	E_{s_3}
0.01	3	-0.277803	0.591567	1.147174	0.149923	0.271322	0.570118	0.354887
0.004	4	-0.295251	0.591559	1.182061	0.134984	0.108209	0.219422	0.130114
0.001	5	-0.296765	0.596736	1.190266	0.128517	0.027022	0.053820	0.030990
0.0001	9	-0.297646	0.595943	1.191234	0.128559	0.002701	0.005352	0.003055
0.000001	27	-0.297753	0.595716	1.191223	0.128694	2.7 e-05	5.35 e-05	3.05 e-05
actual		-0.297754	0.595715	1.191222	0.128696			

Table 1: The stability quantities τ and s_i and their errors $E_\tau, E_{s_{12}}$ and E_{s_3} for different values of local truncation error ϵ. The maximum number of steps required to keep the entries of C within ϵ of their actual values is denoted by m.

5 Concluding remarks

Using symmetry reductions, we have shown how the stability of the figure-eight depends on the first twelfth of its orbit, the piece for which the action is minimized. It seems clear that these reductions should be applicable to other types of symmetric periodic solutions in the n-body problem including choreographies, solutions generated via variational methods with symmetry constraints and symmetric relative equilibria (for example, the kite central configurations of the four-body problem). What is not clear is whether there exists a connection between being a minimizer over a *portion* of the solution and the linear stability of the full orbit. This analysis is challenging for systems with more than two degrees of freedom (compare with Birkhoff's seminal work in [2] showing that minimizing orbits in Hamiltonian systems with two degrees of freedom are always unstable).

In general, periodic solutions which arise as minimizers of the action may be hyperbolic or elliptic. In [18], Offin shows the existence of minimizing periodic orbits in the isosceles three-body problem (two degrees of freedom) which are hyperbolic. These solutions are minimizers over a portion of their orbit. In contrast, Arnaud has shown that for any homotopy class of $\mathbb{T}^n$, there exists an open set of Lagrangians on $T\mathbb{T}^n$ for which the minimizing periodic solution has hyperbolic dimension (the number of eigenvalues off the unit circle) at most two [1]. If n is odd, these solutions are linearly stable. Thus it appears that only for systems with two degrees of freedom is being a minimizer an obstruction to stability. However, the question is far from settled.

Moreover, it would be interesting to have results linking variational techniques with classical stability calculations. For example, is it possible to use variational methods to derive the well-known stability inequality

$$\frac{m_1m_2 + m_1m_3 + m_2m_3}{(m_1+m_2+m_3)^2} < \frac{1}{27}$$

for the Lagrange equilateral triangle solution? We hope to explore some of these interesting questions in future work.

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