Saari's Conjecture for the Restricted Three-body Problem

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Abstract

Saari's conjecture adapted to the restricted three-body problem is proven analytically using BKK theory. Specifically, we show that it is not possible for a solution of the planar, circular, restricted three-body problem to travel along a level curve of the amended potential function unless it is fixed at a critical point (one of the five libration points.) Due to the low dimension of the problem, our proof does not rely on the use of a computer.

MSC Classifications: 70F10, 70F15, 37N05

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1 Introduction

Finiteness questions in the field of celestial mechanics have been prevalent in recent literature. In particular, excellent progress has transpired concerning Saari's conjecture for the classical n-body problem [3, 7, 9, 10], a generalized Saari's conjecture [5, 11, 13] and the Smale/Wintner question [4] concerning the finiteness of relative equilibria in the n-body problem [14, 15].

Saari's conjecture [12] simply states that the only solutions to the n-body problem with a constant moment of inertia (a constant total size) are relative equilibria (rigid rotations). Although at first glance this does not appear to be a question concerning finiteness, one successful approach is to use the mutual distances as variables and show that imposing the fixed inertia constraint on a solution leads to a set of polynomial equations in these variables that has a finite set of solutions. If all of the mutual distances are fixed, then the constant moment of inertia solution must indeed be a rigid rotation, thus proving the conjecture.

This is precisely the approach taken by Rick Moeckel to prove Saari's conjecture for the planar, three-body problem in a wonderful paper utilizing Bernstein-Khovanskii-Kushnirenko (BKK) theory [9]. What makes BKK theory so appealing is that it provides a relatively straight-forward criterion for determining if a system of polynomial equations has a finite number of solutions for which all variables are nonzero. Since both Saari's conjecture and the Smale/Wintner question can be formulated in terms of mutual distances that physically speaking, never vanish, BKK theory is ideally suited to address these challenging problems.

Inspired by Moeckel's work in [9], we use BKK theory to prove a version of Saari's conjecture for the planar, circular, restricted three-body problem. We show that the only solutions with a constant value of the amended potential are equilibria, critical points of the potential function. Equivalently, the only solutions with a constant velocity are fixed points (velocity zero). After motivating the problem, we derive two polynomial equations in two distance variables which must be satisfied by a solution with constant potential. We then apply BKK theory to show that the number of solutions to this system is finite, thereby obtaining the result. Due to the low dimension of the problem, it is possible to do all the necessary calculations by hand.

2 The Restricted Three-body Problem

To begin, we examine Saari's conjecture for the Newtonian n-body problem [12]. Suppose that $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n) \in \mathbb{R}^{2n}$ represents the coordinates of n bodies in the plane with respective masses m_i. The classical, planar n-body problem is characterized by the Newtonian potential function U given by

$$U(\mathbf{q}) = \sum_{i<j} \frac{m_i m_j}{r_{ij}}$$

where $r_{ij} = ||\mathbf{q}_i - \mathbf{q}_j||$ measures the distance between the i-th and j-th bodies. The equations of motion for the i-th body are

$$m_i \ddot{\mathbf{q}}_i = \frac{\partial U}{\partial \mathbf{q}_i}.$$

The corresponding set of differential equations is Hamiltonian with Hamiltonian $H = K - U$, where K is the kinetic energy, given by

$$K(\dot{\mathbf{q}}) = \frac{1}{2} \sum_{i=1}^{n} m_i ||\dot{\mathbf{q}}_i||^2.$$

The moment of inertia I measures the total size of the system,

$$I(\mathbf{q}) = \frac{1}{2} \sum_{i=1}^{n} m_i \|\mathbf{q}_i\|^2.$$

Without loss of generality, we have taken the moment of inertia with respect to a center of mass fixed at the origin.

A solution rigidly rotating about its center of mass (the origin in our case) is called a *relative equilibrium* because it is a fixed point in a rotating coordinate system. This special solution takes the form $\mathbf{q}_i = R(\omega t)\mathbf{x}_i$ where

$$R(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

is a rotation matrix and x_i is the initial position of the i-th body. Clearly, such a solution has a constant moment of inertia. Saari conjectured that these are the only solutions in the n-body problem having constant I [12].

Imposing the constraint $I = c$ on a solution is more restrictive than it first appears due to the Lagrange-Jacobi identity [6]. Differentiating I twice with respect to t and using the fact that U is a homogeneous function of degree -1 leads to

$$\ddot{I} \;=\; 2K - U \;=\; 2H + U.$$

Since H is a constant of motion, any solution with constant moment of inertia I must also have a constant value of U and thereby, a constant value of K as well. This fact helps us motivate a related conjecture for the restricted three-body problem.

The planar, circular, restricted three-body problem (PCR3BP) consists of two large masses traveling along circular orbits and a third infinitesimal mass subject to the gravitation of the two large "primaries." All three bodies are traveling in the same plane of motion and the infinitesimal mass is assumed to have no effect on the motion of the larger bodies. Without loss of generality, we take the total mass of the two large primaries to be one. The equations of motion for the third mass are usually taken in a revolving frame, rotating at the same speed as the two primaries. In this frame, the first primary has position $\mathbf{q}_1 = (1-\mu, 0)$ and mass $m_1 = \mu$ while the second is located at $\mathbf{q}_2 = (-\mu, 0)$ with mass $m_2 = 1-\mu$. Note that the center of mass of the two primaries is at the origin and consequently, in the non-rotating frame, each primary is on a circular orbit centered at the origin. The mass parameter μ is usually chosen so that $0 < \mu < 1$, although our methods allow for any value of μ (even complex) with the important and physically necessary condition that $\mu \neq 0, 1$.

Let (x, y) denote the position of the third body in the rotating frame with velocity (u, v). The variables we use in our two key polynomial equations are a and b, representing the distances from the infinitesimal particle to the first and second primary, respectively. Specifically, we have

$$a = \sqrt{(x-1+\mu)^2 + y^2} \quad \text{and} \quad b = \sqrt{(x+\mu)^2 + y^2}. \tag{1}$$

The values of a and b determine a unique position (x, y) of the infinitesimal mass up to a sign choice of y. The inverse of equation (1) is given by

$$x = \tfrac{1}{2} - \mu + \tfrac{1}{2}(b^2 - a^2) \quad \text{and} \quad y = \pm\tfrac{1}{2}\sqrt{-a^4 + 2a^2b^2 + 2a^2 - b^4 + 2b^2 - 1}. \tag{2}$$

The equations of motion describing the trajectory of the infinitesimal mass in the rotating frame are given by

$$\begin{aligned}\dot{x} &= u \\ \dot{y} &= v \\ \dot{u} &= V_x + 2v \\ \dot{v} &= V_y - 2u\end{aligned} \tag{3}$$

where

$$V(x,y) = \frac{1}{2}(x^2+y^2) + \frac{\mu}{a} + \frac{1-\mu}{b} + \frac{1}{2}\mu(1-\mu)$$

is called the *amended potential.* It is well-known and easy to check that the quantity

$$E = \frac{1}{2}(u^2+v^2) - V \tag{4}$$

is conserved for the above system. This is essentially the Jacobi integral. We have added a constant to V to make the change of variables from (x,y) to (a,b) easier. Regardless of the value of the velocity (u,v), we must have $E+V \geq 0$. Thus, for a fixed value of the energy integral E, the inequality $V(x,y) \geq -E$ determines a region in the xy-plane for which the motion of the particle is confined.

The question we are concerned with in this work is whether it is possible for a solution of the PCR3BP to travel along a level curve of V. By equation (4), such a solution would necessarily have constant velocity (constant kinetic energy). Thus, this condition imposes the same constraints as does Saari's conjecture in the full n-body problem.

Moreover, any equilibrium in the PCR3BP has a constant value of V. In fact, the equilibria for system (3) are precisely the critical points of V. It is well-known that there are exactly five such points, called the *libration points* in classical literature [8]. Two of these are minima located at $(1/2-\mu, \pm\sqrt{3}/2)$, each forming an equilateral triangle configuration with the two primaries. The remaining critical points are saddles found along the x-axis, one in each of the three open intervals determined by the primaries, $(-\infty,-\mu), (-\mu, 1-\mu)$ and $(1-\mu,\infty)$. Our main result states that these are the only solutions with constant V, thus proving a version of Saari's conjecture for the restricted three-body problem.

Theorem 2.1 *The only solutions to the circular, planar, restricted three-body problem with a constant value of the amended potential V are equilibria.*

3 Saari's Conjecture

In this section we derive two polynomial equations in a and b that must be satisfied by any solution having a constant value of the amended potential. We then apply a theorem from BKK theory to prove the number of solutions to these equations is finite. By equations (1) and (2), the number of solutions to any system of equations in one set of variables is finite if and only if it is finite in the other set of variables. Thus, the values of (x,y) being held constant implies that the solution is fixed and therefore at equilibrium.

To obtain two equations in two unknowns we eliminate the velocity variables u and v. Suppose that $(x(t), y(t))$ is a solution to the PCR3BP for which $V(x(t), y(t)) = c/2$ is constant. As mentioned above, a constant value of V forces a constant value of the velocity. We set $u^2+v^2 = k$ for some nonnegative constant k. Differentiating $V = c/2$ with respect to t gives the identity $V_x u + V_y v = 0$ which can then be used to solve for u and v,

$$u = \frac{\pm\sqrt{k}\, V_y}{||\nabla V||} \quad \text{and} \quad v = \frac{\mp\sqrt{k}\, V_x}{||\nabla V||}.$$

The signs indicate that if $+$ is chosen for u, then $-$ is taken for v and vice-versa. Our final equation comes from differentiating $V = c/2$ with respect to t twice. By substituting in the expressions for u and v and using the equations of motion, the end result is two equations in x and y. In sum, the system of four equations in the variables x, y, u and v

$$\begin{aligned} V &= c/2 \\ u^2 + v^2 &= k \\ V_x u + V_y v &= 0 \\ \ddot{V} &= 0 \end{aligned}$$

can be reduced to a system of two equations in the variables x and y,

$$\begin{aligned} V &= c/2 \\ ||\nabla V||^4 \mp 2\sqrt{k}\, ||\nabla V||^3 + k(V_x^2 V_{yy} - 2V_x V_y V_{xy} + V_y^2 V_{xx}) &= 0. \end{aligned} \tag{5}$$

Denote $\Lambda = V_x^2 V_{yy} - 2V_x V_y V_{xy} + V_y^2 V_{xx}$. Equation (5) has an interesting geometric interpretation in terms of curvature. Using the formula for the curvature of a planar curve, any solution to the differential equation $\dot{x} = \pm V_y, \dot{y} = \mp V_x$ (necessarily traveling along a level curve of V) will have curvature

$$\kappa = \mp \frac{\Lambda}{||\nabla V||^3}.$$

This gives an intrinsic expression for the curvature of a level curve of V at the point (x, y). On the other hand, our solution $(x(t), y(t))$ traveling along a level curve of V has curvature

$$\kappa = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{k^{3/2}}.$$

Equating these two expressions for curvature and using $u^2+v^2 = k$ gives an alternative derivation of equation (5).

To remove the square roots from the second term on the left-hand side of equation (5), we rearrange terms and square both sides. Our final system is

$$V = c/2 \tag{6}$$

$$||\nabla V||^8 - 4k||\nabla V||^6 + 2k\Lambda||\nabla V||^4 + k^2\Lambda^2 = 0. \tag{7}$$

Converting these equations into the variables a and b and clearing denominators produces a system of two polynomial equations.

Equation (6) is readily converted into ab-coordinates using the identity

$$x^2 + y^2 = \mu a^2 + (1-\mu)b^2 + \mu^2 - \mu. \tag{8}$$

Using this substitution and multiplying through by ab, we obtain our first polynomial equation,

$$\mu a^3 b + (1-\mu)ab^3 - cab + 2(1-\mu)a + 2\mu b = 0. \tag{9}$$

To facilitate the calculations, we have written the polynomial on the left-hand side in lexicographic order with the ordering $a > b$.

Equation (7) is considerably harder to compute by hand. Indeed, calculations using Maple show that, upon clearing denominators, this polynomial has 404 terms (monomials in $\mathbb{C}[a,b]$). When the coefficients are expanded, it requires 30 8.5×11 pages to render the entire polynomial. Fortunately, as explained in the next subsection, we need only compute two terms inside equation (7), $||\nabla V||^2$ and Λ, both of which can be calculated by hand.

Taking partial derivatives of V, we find

$$\begin{aligned}
V_x &= x - \frac{\mu(x-1+\mu)}{a^3} - \frac{(1-\mu)(x+\mu)}{b^3} \\
V_y &= y\left(1 - \frac{\mu}{a^3} - \frac{(1-\mu)}{b^3}\right) \\
V_{xx} &= 1 - \frac{\mu(y^2 - 2(x-1+\mu)^2)}{a^5} - \frac{(1-\mu)(y^2 - 2(x+\mu)^2)}{b^5} \\
V_{yy} &= 2 + \frac{\mu}{a^3} + \frac{1-\mu}{b^3} - V_{xx} \\
V_{xy} &= 3y\left(\frac{\mu(x-1+\mu)}{a^5} + \frac{(1-\mu)(x+\mu)}{b^5}\right).
\end{aligned}$$

Using equations (2), (8) and the identity

$$(x+\mu)(x-1+\mu) + y^2 = \frac{1}{2}(a^2 + b^2 - 1),$$

we compute

$$\begin{aligned}
||\nabla V||^2 = (&\mu a^6 b^4 - \mu(1-\mu)a^6 b + (1-\mu)a^4 b^6 - \mu(1-\mu)a^4 b^4 - (2-\mu)(1-\mu)a^4 b^3 \\
&+\mu(1-\mu)a^4 b + (1-\mu)^2 a^4 - \mu(1+\mu)a^3 b^4 + \mu(1-\mu)a^3 b - \mu(1-\mu)ab^6 \\
&+\mu(1-\mu)ab^4 + \mu(1-\mu)ab^3 - \mu(1-\mu)ab + \mu^2 b^4)/a^4 b^4.
\end{aligned}$$

The calculation of Λ, involving both the first and second partials of V, is more laborious, resulting in an expression containing 44 terms in the numerator (see the Appendix for details.)

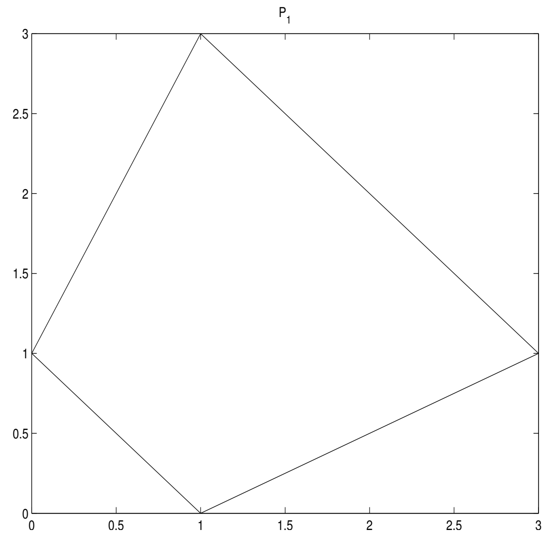

Figure 1: The Newton polytope for the equation $\mu a^3 b + (1-\mu)ab^3 - cab + 2(1-\mu)a + 2\mu b = 0$ is a trapezoid in the ab-plane.

3.1 BKK Theory

To prove our main result we use a theorem from BKK theory first introduced to the field by Moeckel [9]. Let $\mathbb{C}^* = \mathbb{C} - \{0\}$. We think of each element $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n_{\geq 0}$ as an exponent vector of the monomial $z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}$, abbreviated simply as z^k. A polynomial $f \in \mathbb{C}[z_1, \ldots z_n]$ is a sum of monomials, generically written as

$$f = \sum_k c_k z^k$$

where $c_k \in \mathbb{C}^*$ for each k and there are only a finite number of terms in the sum. The Newton polytope for f, denoted $N(f)$, is the convex hull in $\mathbb{R}^n$ of the set of all exponent vectors k occurring for f. For example, the exponent vectors for the polynomial in equation (9) are $(3,1), (1,3), (1,1), (1,0), (0,1)$. Therefore, the Newton polytope for this polynomial is a trapezoid in the ab-plane, with vertices at $(3,1), (1,3), (1,0), (0,1)$. The vertex $(1,1)$ lies inside the convex hull determined by the others (see Figure 1).

Suppose that $r = (r_1, r_2, \ldots, r_n)$ is a solution to the system of m polynomial equations

$$\begin{aligned} f_1(z_1, \ldots, z_n) &= 0 \\ f_2(z_1, \ldots, z_n) &= 0 \\ &\vdots \\ f_m(z_1, \ldots, z_n) &= 0, \end{aligned} \tag{10}$$

that is, r belongs to the affine variety $\mathbb{V}(f_1, \ldots, f_m)$. We say that r lies in the *algebraic torus* $\mathbb{T}$ if $r_i \in \mathbb{C}^*$ $\forall i$. For our problem, since neither variable can vanish, we are only interested in the part of the variety lying in $\mathbb{T}$. In general, we will call r a *trivial solution* if $r_i = 0$ for some i.

The concept of the *reduced equation* is central to our work. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a vector of rational numbers. For a given polynomial f, the *reduced polynomial* f_α is the sum of all terms of f whose exponent vectors k satisfy

$$\alpha \cdot k = \min_{l \in N(f)} \alpha \cdot l. \tag{11}$$

The last equation defines a face of the polytope $N(f)$ with inward pointing normal α, although this face is not necessarily of codimension one. For all exponent vectors k on this face, the dot product $\alpha \cdot k$ will be strictly smaller than the dot product of α with any exponent vector elsewhere in $N(f)$.

Continuing our example above, consider the rational vector $\alpha = (1,1)$ and the Newton polytope determined by the polynomial from equation (9). This is the inward pointing normal for the lower left face and clearly, $\alpha \cdot k$ attains a minimum of one along this face. Geometrically speaking, the family of parallel lines $a + b = \gamma$, determined by the normal vector $\alpha = (1,1)$, attains a minimum value of $\gamma = 1$ over the Newton polytope $N(f)$. The only exponent vectors coming from condition (11) are $(1,0)$ and $(0,1)$. Therefore, for $\alpha = (1,1)$, the reduced polynomial f_α is simply

$$f_\alpha(a,b) = 2(1-\mu)a + 2\mu b.$$

In contrast, suppose that $\alpha = (1,3)$. This is not a normal vector to any of the four sides of the trapezoid. However, the mimimum of $\alpha \cdot k$ is attained over $N(f)$ at the vertex $(1,0)$. This is a "face" of codimension 2. Geometrically speaking, the family of parallel lines $a + 3b = \gamma$ taken over $N(f)$ has a mimimum value of $\gamma = 1$ at the corner point $(1,0)$. The reduced polynomial in this case is simply

$$f_\alpha(a,b) = 2(1-\mu)a$$

since the only exponent vector satisfying condition (11) is $(1,0)$.

For a given rational vector $\alpha = (\alpha_1, \ldots, \alpha_n)$, the reduced equations for system (10) are defined using the reduced polynomials corresponding to α:

$$\begin{aligned} f_{1\alpha}(z_1, \ldots, z_n) &= 0 \\ f_{2\alpha}(z_1, \ldots, z_n) &= 0 \\ &\vdots \\ f_{m\alpha}(z_1, \ldots, z_n) &= 0. \end{aligned} \tag{12}$$

Bernstein makes use of these reduced equations in the following theorem [1]. A readable proof using algebraic geometry and fractional-power Puiseux series can be found in [9].

Theorem 3.1 *Suppose that system (10) has infinitely many solutions in* $\mathbb{T}$. *Then there exists a vector* $\alpha = (\alpha_1, \ldots, \alpha_n)$ *with* $\alpha_i \in \mathbb{Q}$ *and* $\alpha_j = 1$ *for some* j, *such that the system of reduced equations (12) also has a solution in* $\mathbb{T}$.

We will apply Theorem 3.1 to the polynomial system determined by equations (6) and (7), written as polynomials in $\mathbb{C}[a,b]$. For notational convenience, let

$$A = a^4 b^4 ||\nabla V||^2 \quad \text{and} \quad B = 4a^7 b^7 \Lambda$$

so that $A, B \in \mathbb{C}[a,b]$. We clear denominators in equation (7) by multiplying through by $16a^{16}b^{16}$. This yields the polynomial system

$$\begin{aligned} \mu a^3 b + (1-\mu)ab^3 - cab + 2(1-\mu)a + 2\mu b &= 0 & (13) \\ 16A^4 - 64ka^4b^4A^3 + 8kabA^2B + k^2a^2b^2B^2 &= 0. & (14) \end{aligned}$$

As mentioned above, the Newton polytope for the first polynomial is a trapezoid in the ab-plane, with vertices at $(3,1),(1,3),(1,0),(0,1)$. To compute the second Newton polytope without actually expanding each of the four terms in equation (14), we just search for the "outermost" exponent vectors in $\mathbb{Z}^2_{\geq 0}$. Any exponent vector (k_1, k_2) lying vertically between two other exponent vectors, can be ignored. For example, in A, the exponent vectors $(1,4)$ and $(1,3)$ lie between $(1,6)$ and $(1,1)$. Equivalently,

$$ab^6 >_{lex} ab^4 >_{lex} ab^3 >_{lex} ab.$$

For a fixed power of a, we only keep the vectors with the highest and lowest powers in b, since everything in between will lie inside the Newton polytope. This gives a simple method for computing the Newton polytope of a polynomial in two variables. For each power of a, delete any exponent vectors lying vertically between two others. Then, from the remaining set of vectors, throw out any that lie inward of a line segment connecting two outer vectors.

An added benefit of this method is that we can construct the Newton polytope of A^4 by building up from the outermost exponent vectors of lower powers of A. This follows because lexicographical ordering (or any monomial ordering) is preserved under multiplication by a monomial. The "inner" vectors will only contribute to create more inner vectors and are thus ignorable when computing the polytope of higher powers. For example, the outermost exponent vectors of A are

$$(6,4),(6,1),(4,6),(4,0),(3,4),(3,1),(1,6),(1,1),(0,4)$$

while the ignorable, inner vectors are

$$(4,4),(4,3),(4,1),(1,4),(1,3).$$

Consequently, the outermost exponent vectors of A^2 are

$$\begin{gathered}(12,8),(12,2),(10,10),(10,1),(9,8),(9,2),(8,12),(8,0),(7,10),(7,1),(6,8),(6,2),(5,12),(5,1),\\(4,10),(4,2),(3,8),(3,5),(2,12),(2,2),(1,10),(1,5),(0,8)\end{gathered}$$

all of which are generated by adding two outermost exponent vectors of A together. Continuing in this fashion, the outermost vectors for A^4 are easily deduced to be

$$\begin{gathered}(24,16),(24,4),(22,18),(22,3),(21,16),(21,4),(20,20),(20,2),(19,18),(19,3),(18,22),(18,1),\\(17,20),(17,2),(16,24),(16,0),(15,22),(15,1),(14,20),(14,2),(13,24),(13,1),(12,22),(12,2),\\(11,20),(11,3),(10,24),(10,2),(9,22),(9,3),(8,20),(8,4),(7,24),(7,3),(6,22),(6,4),\\(5,20),(5,7),(4,24),(4,4),(3,22),(3,7),(2,20),(2,10),(1,18),(1,13),(0,16).\end{gathered}$$

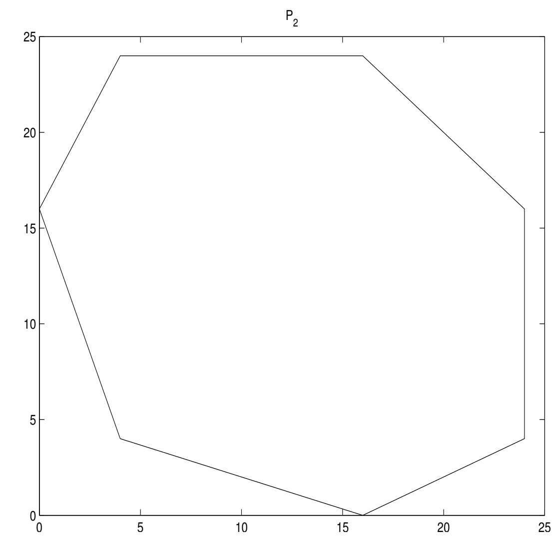

Figure 2: The Newton polytope for equation (14) has only seven vertices.

The outermost vectors for the remaining three terms of equation (14), $a^4b^4A^3$, abA^2B and $a^2b^2B^2$, can be computed in a similar manner. Taking the union of all these vectors and then computing the outermost vectors of that union gives

$$(24,16),(24,4),(22,18),(22,3),(21,16),(21,4),(20,20),(20,2),(19,18),(19,3),(18,22),(18,1),$$
$$(17,20),(17,2),(16,24),(16,0),(15,22),(15,1),(14,20),(14,2),(13,24),(13,1),(12,22),(12,2),$$
$$(11,24),(11,2),(10,24),(10,2),(9,22),(9,3),(8,24),(8,3),(7,24),(7,3),(6,24),(6,4),$$
$$(5,24),(5,4),(4,24),(4,4),(3,22),(3,7),(2,20),(2,10),(1,18),(1,13),(0,16).$$

Remarkably, most of these exponent vectors come from the first term in equation (14), $16A^4$. Only six vectors come from $8kabA^2B$, two arise out of $k^2a^2b^2B^2$ and none come from $-64ka^4b^4A^3$. A simple hand-drawn plot of these last 47 vectors yields the Newton polytope for the polynomial in equation (14). It has only seven vertices:

$$(24,16),(24,4),(16,24),(16,0),(4,24),(4,4),(0,16)$$

(see Figure 2). Although this seems a surprisingly small number, many of the remaining exponent vectors (22 to be precise) can not be discarded as they lie on an edge of the Newton polytope and consequently will have to be included in the reduced equations for that face.

3.2 Proof of Theorem 2.1

We now prove Theorem 2.1 using BKK theory. By contradiction, suppose that the system of equations (13) and (14) has an infinite number of solutions in $\mathbb{T}$. By Theorem 3.1, there exists a vector of rationals (α_1, α_2) with $\alpha_i = 1$ for some i such that the system of reduced equations also has a solution in $\mathbb{T}$. We will show that all choices of α lead to only trivial solutions, where either $a = 0$ or $b = 0$. Throughout the proof, the only restrictions on the mass parameter μ are $\mu \neq 0, \mu \neq 1$.

First, consider the Newton polytope for the simpler equation (13). Moving counterclockwise from the top of the polytope, the inward pointing normals with at least one component equal to one are $(1, -1/2), (1, 1)$ and $(-1/2, 1)$. We claim that these are the only three vectors we need to consider. Each vertex of the polytope corresponds to only one term of the polynomial, and this term vanishes only if one of a or b is zero. For example, any vector of the form $\alpha = (1, q)$, where q is some rational number satisfying $-1/2 < q < 1$, will achieve a minimum dot product over the polytope at the vertex $(0, 1)$. The first reduced equation for all of these vectors is identical, simply $2\mu b = 0$ which has no solutions in $\mathbb{T}$. In this case, we avoid having to examine the reduced equation of (14) because the first reduced equation has only trivial solutions. Thus, even though $(1, 1/3)$ is an inward pointing normal for the second polytope, it leads to a trivial reduced first equation. A similar argument works for the other three vertices, thereby eliminating all vectors α from contention other than $(1, -1/2), (1, 1)$ and $(-1/2, 1)$.

Beginning with $\alpha = (1, -1/2)$, we see that this is also an inward pointing normal for the second polytope. The exponent vectors lying on this edge for the second polytope are $(0, 16), (1, 18), (2, 20), (3, 22), (4, 24)$ all of which come solely from the A^4 term in equation (14). In fact, all of these exponent vectors arise from raising the binomial $-\mu(1-\mu)ab^6 + \mu^2 b^4$ to the fourth power. Thus, the reduced equations corresponding to $\alpha = (1, -1/2)$ are

$$\begin{aligned} (1-\mu)ab^3 + 2\mu b &= 0 \\ 16(-\mu(1-\mu)ab^6 + \mu^2 b^4)^4 &= 0 \end{aligned}$$

which simplifies to

$$\begin{aligned} b((1-\mu)ab^2 + 2\mu) &= 0 \\ 16\mu^4 b^{16}(-(1-\mu)ab^2 + \mu)^4 &= 0. \end{aligned}$$

Since $b \neq 0$, we can substitute $-(1-\mu)ab^2 = 2\mu$ from the first equation into the second to obtain

$$16\mu^4 b^{16}(3\mu)^4 = 0.$$

Since this last equation has only $b = 0$ as a solution (assuming $\mu \neq 0$), we have eliminated $\alpha = (1, -1/2)$ from contention.

Next, we consider the vector $\alpha = (1, 1)$ which leads to a nontrivial reduced first equation. However, the reduced second equation for this vector corresponds to just the vertex $(4, 4)$ and like the previous case, this exponent vector arises from the A^4 term. This gives a trivial reduced second equation of

$$16(-\mu(1-\mu)ab)^4 = 0$$

which has no solutions in $\mathbb{T}$ since $\mu(1-\mu) \neq 0$.

Finally, we examine the reduced equations for the vector $\alpha = (-1/2, 1)$. This is also an inward pointing normal of the second polytope, whose edge contains the exponent vectors $(24, 4), (22, 3), (20, 2), (18, 1)$ and $(16, 0)$. Fortunately, as with $\alpha = (1, -1/2)$, these exponents all come from the A^4 term, occurring by raising the binomial $-\mu(1-\mu)a^6b + (1-\mu)^2a^4$ to the fourth power. In factored form, the reduced equations for $\alpha = (-1/2, 1)$ are

$$\begin{aligned} a(\mu a^2 b + 2(1-\mu)) &= 0 \\ 16(1-\mu)^4 a^{16}(-\mu a^2 b + 1 - \mu)^4 &= 0. \end{aligned}$$

Since $a \neq 0$, we can substitute $-\mu a^2 b = 2(1-\mu)$ into the second equation, obtaining

$$16(1-\mu)^4 a^{16}(3(1-\mu))^4 = 0.$$

This last equation has only $a = 0$ as a solution since $\mu \neq 1$. This eliminates the last remaining vector from contention. Thus, for any choice of α, the reduced equations do not have a solution in $\mathbb{T}$. This contradiction proves that the system of equations (13) and (14) has a finite number of nonzero solutions. This completes the proof of Theorem 2.1. □

Remarks:

1. As stated in the introduction, a simple corollary to Theorem 2.1 is that any solution to the PCR3BP with a constant velocity must be at rest at one of the five libration points. This follows from equation (4) since constant velocity $u^2 + v^2$ implies constant potential V. This has an interesting interpretation in terms of orbital mechanics. For example, it is not possible to have a satellite, moving only under the influence of gravity, to orbit the Earth-moon system at a constant speed.

2. In [9], Moeckel computes the normal fan of the Minkowski sum polytope (the "big Minkowski") in order to insure that all vectors α are accounted for when applying Theorem 3.1. In our case, this is not necessary since we only have two polytopes and the reduced first equation is trivial at each vertex of the first polytope.

3. One of the nice aspects of doing the calculations by hand for equation (14) is that it provides insight into why Theorem 3.1 can be successfully applied to our problem. In the calculation of the second Newton polytope, nearly all of the vertices arise from the A^4 term. Consequently, the reduced second equation for the inward pointing normals come from raising a positive term to the fourth power. Thus it is not surprising that such an equation has only trivial solutions. Interestingly enough, using Theorem 3.1 to show that the number of equilibrium points for system (3) is finite will not succeed. In this case, the reduced equations for $V_x = 0, V_y = 0$ written as polynomials in a and b have nontrivial solutions along two faces.

4. The area of the Minkowski sum polytope is 556.5. Subtracting off the areas of the polytopes in Figures 1 and 2 gives a mixed volume (or area in this case) of 104. This is the number of nontrivial solutions to equations (13) and (14) counted with multiplicities. Since the number of real solutions is 4 (one for the Lagrange equilateral triangle $a = b = 1$, and three for the collinear equilibria), 100 of these solutions must be complex-valued.

4 Appendix

To calculate $\Lambda = V_x^2 V_{yy} - 2V_x V_y V_{xy} + V_y^2 V_{xx}$ by hand, we simplified the quantity $V_x^2 V_{yy} + V_y^2 V_{xx}$ and $-2V_x V_y V_{xy}$ separately, converted each expression into a and b variables, and then combined like terms, obtaining

$$\begin{gathered}
\Lambda = (-3\mu^2(1-\mu)a^{11}b^2 + 4\mu a^9 b^7 - 2\mu(1-\mu)(4-3\mu)a^9 b^4 + 6\mu^2(1-\mu)a^9 b^2 \\
+4\mu(1-\mu)^2 a^9 b + 6\mu^2(1-\mu)a^8 b^2 + 4(1-\mu)a^7 b^9 - 4\mu(1-\mu)a^7 b^7 - (1-\mu)(3\mu^2 - 8\mu + 12)a^7 b^6 \\
+2\mu(1-\mu)(4+\mu)a^7 b^4 + 4(1-\mu)^2(3-\mu)a^7 b^3 - 3\mu^2(1-\mu)a^7 b^2 - 4\mu(1-\mu)^2 a^7 b - 4(1-\mu)^3 a^7 \\
-\mu(3\mu^2 + 2\mu + 7)a^6 b^7 + 2\mu(1-\mu)(7-5\mu)a^6 b^4 - 12\mu^2(1-\mu)a^6 b^2 - 7\mu(1-\mu)^2 a^6 b \\
-3\mu^2(1-\mu)a^5 b^2 - 2\mu(1-\mu)(1+3\mu)a^4 b^9 + 2\mu(1-\mu)(5-\mu)a^4 b^7 + 2\mu(1-\mu)(2+5\mu)a^4 b^6 \\
-20\mu(1-\mu)a^4 b^4 - 2\mu(1-\mu)^2 a^4 b^3 + 6\mu^2(1-\mu)a^4 b^2 + 10\mu(1-\mu)^2 a^4 b + 4\mu^2(2+\mu)a^3 b^7 \\
-2\mu^2(1-\mu)a^3 b^4 + 6\mu^2(1-\mu)a^3 b^2 - 3\mu(1-\mu)^2 a^2 b^{11} + 6\mu(1-\mu)^2 a^2 b^9 + 6\mu(1-\mu)^2 a^2 b^8 \\
-3\mu(1-\mu)^2 a^2 b^7 - 12\mu(1-\mu)^2 a^2 b^6 - 3\mu(1-\mu)^2 a^2 b^5 + 6\mu(1-\mu)^2 a^2 b^4 + 6\mu(1-\mu)^2 a^2 b^3 \\
-3\mu(1-\mu)^2 a^2 b + 4\mu^2(1-\mu)ab^9 - 4\mu^2(1-\mu)ab^7 - 7\mu^2(1-\mu)ab^6 + 10\mu^2(1-\mu)ab^4 \\
-3\mu^2(1-\mu)ab^2 - 4\mu^3 b^7)/(4a^7 b^7).
\end{gathered}$$

This expression was confirmed using Maple.

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