Research Statement

Since receiving tenure, my research has primarily been focused in two areas: celestial mechanics and the n-vortex problem. I have also done some work studying numerical methods from a dynamical systems perspective. Conducting research with undergraduates continues to play an important role in my scholarship. Each of these areas is described below, with a focus on the work I have published since receiving tenure.

1. Celestial Mechanics

The subject of celestial mechanics concerns the motion of n celestial bodies (stars, planets, asteroids, spaceships, etc.) interacting solely under their mutual gravitational attraction. Here n is a whole number larger than one. This is the classical, Newtonian n-body problem and is described by a complicated system of nonlinear differential equations.

The problem is well-understood for the case $n = 2$, a two-body problem such as the sun and the Earth, often referred to as the *Kepler problem.* In this case, a small mass orbits a larger body, its trajectory tracing out the path of a conic section (circle, ellipse, parabola, or hyperbola). For over 300 years mathematicians and scientists have struggled to understand the complicated structure of the n-body problem for n at least three. For instance, in attempting to explain known anomalies in the motion of the Earth's moon, Newton investigated the three-body problem by incorporating the gravitational effects of the sun, but was so distraught he confided to a colleague that "his head never ached but with his study on the moon."[1] On the other hand, research in the n-body problem has enabled us to fly to the moon, place satellites in orbit around the Earth, design affordable spacecraft missions to explore the solar system, and locate interesting celestial orbits.

My recent research in the n-body problem has focused on understanding the properties of central configurations, an important and active subfield of celestial mechanics. I have also investigated the linear stability of some special periodic orbits that contain collisions.

Central configurations

A special configuration of bodies, where the gravitational force on each body points toward the center of mass, is called a *central configuration.* If the bodies in a central configuration have zero initial velocity (starting from rest), then the ensuing motion will be a contraction of the original configuration, resulting in total collision (a homothetic motion). Throughout the orbit, the configuration maintains its initial shape. For example, placing three bodies at the vertices of an equilateral triangle, with any choice of masses, yields a central configuration of the three-body problem. Starting from rest, the orbit will always appear as an equilateral triangle, although the size of the triangle will decline toward zero.

Central configurations were some of the earliest solutions discovered in the n-body problem. The equilateral triangle solution was found by Lagrange in 1772 and although he was unaware of its significance at the time, twentieth century astronomers later discovered two groups of asteroids (now called the Trojans and the Greeks) forming an equilateral triangle with the sun

[1] Westfall, *Never at Rest: A Biography of Isaac Newton*, p. 541.

and Jupiter. Moreover, a recent asteroid, 2010 $\mathrm{T}K_7$, has been discovered by Canadian scientists to be located on average at a "Trojan point" in the Earth-Sun system, lying at a 60-degree angle from the Earth [5]. Central configurations also play a significant role in astronomy and spacecraft mission design because they provide useful information on nearby orbits, information that has been used by scientists to design inexpensive space missions [10].

Any planar central configuration leads to a one-parameter family of solutions in the n-body problem where each body is traveling along a particular Kepler orbit. Included in this family are important periodic solutions where the bodies orbit the center of mass on circles. In this case the motion is simply a rigid rotation, otherwise known as a *relative equilibrium*, since it is a fixed point in rotating coordinates. (Imagine rotating along with the bodies at the same speed, riding on a sort of celestial merry-go-round. From your perspective, the bodies will appear to be fixed.) For a relative equilibrium solution, both the shape and size of the initial configuration is preserved throughout the motion. It is known that only planar central configurations lead to these special circular solutions.

The algebraic system of equations determining a central configuration are nonlinear and quite complicated. The goal, given a set of masses m_i for each body, is to find positions $x_1, \ldots, x_n$ and a constant λ (not depending on i) such that

$$\lambda\, x_i \;=\; \sum_{j \neq i}^{n} \frac{m_j (x_i - x_j)}{r_{ij}^3} \tag{1}$$

is satisfied for each $i \in \{1, \ldots n\}$, where $r_{ij} = ||x_i - x_j||$ denotes the distance between the i-th and j-th bodies.

Given the difficult nature of the problem, one approach is to make restrictions on the initial positions of the bodies, and then proceed to search for solutions within this constraint. Toward that end, working with Josep Cors (Universitat Politècnica de Catalunya, Barcelona, Spain), I studied co-circular central configurations in the four-body problem. A central configuration is *co-circular* if the bodies all lie on a common circle. Using mutual distances as coordinates, we showed that the set of four-body co-circular central configurations with positive masses is a two-dimensional surface, a graph over two of the exterior side-lengths. Two symmetric families, the kite and isosceles trapezoid, were investigated extensively. We also proved that a specific ordering of the masses is required for a four-body co-circular central configuration and obtained explicit bounds on the mutual distances. Interestingly, we showed that if any two masses in a co-circular central configuration are equal, then the configuration is symmetric, either a kite or an isosceles trapezoid. In addition to utilizing many analytic arguments, our techniques employed ideas from classical geometry and modern computational algebra. Our paper, titled "Four-Body Co-Circular Central Configurations," was published in the journal *Nonlinearity* in 2012 [7].

Building on this research, I collaborated with my thesis advisor Glen Hall (Boston University) and Josep Cors to work on an interesting open question on co-circular central configurations. It is clear that the regular n-gon (n equal masses placed at the vertices of a regular n-gon) is a co-circular central configuration. Because of its symmetry, the center of mass of the configuration coincides with the center of the circle containing the bodies. This implies that the ensuing motion will have all n bodies following each other around the *same* circle, a special type of motion known in the field as a *choreography.* Typically this is not the case, as the center of mass is usually distinct from the center of the circumscribing circle. For example, if the configuration has a mass much larger than the others, the center of mass will reside close to this body.

The open question we studied is whether the regular n-gon is the *only* co-circular central configuration having its center of mass located at the center of the circle containing the configuration. This problem has garnered significant attention, listed as Problem 12 in a collection of open problems in the field recently published by Albouy, Cabral, and Santos [1]. Although we initially focused on the n-body case, we soon realized that our techniques worked for a wider range of potential functions, including the n-vortex problem (to be discussed in the next section). Beginning with a central configuration located on the unit circle, one whose center of mass is at the origin, we derived two new sets of equations that are necessary and sufficient for such a configuration to exist. The two sets of equations correspond to angular and radial perturbations of the configuration. Using a topological approach, we showed that for any choice of positive masses, if such a central configuration exists, then it is unique. For the planar n-vortex problem with arbitrary vorticities, we then showed that the only possible solution is the regular n-gon with equal circulations. We also showed that if equal masses are specified, then the regular n-gon is the only possible solution for any of the potential functions considered. In sum, we completely solved the open question for the n-vortex case, and made modest progress toward a resolution for the n-body case. Our paper, titled "Uniqueness Results for Co-Circular Central Configurations for Power-Law Potentials," was published in *Physica D: Nonlinear Phenomena* in 2014 [6].

Another paper on central configurations arose from my work mentoring HC undergraduate researchers Julianne (Julie) Kulevich and Christopher (Chris) Smith (both HC '08). We investigated central configurations in the planar, circular, restricted four-body problem (PCR4BP). In this problem, three large bodies (called *primaries*) of arbitrary mass are located at the vertices of a rotating equilateral triangle while a fourth body (e.g., asteroid or spaceship) with negligible mass is inserted into the same plane of motion as the triangle. The goal is to study the possible motions of the fourth body under the assumption that the three large bodies remain on the rotating triangle. The simplest such motion would be a fixed point, that is, a location where the fourth body is in perfect balance with the gravitational attraction of the primaries. Taken together, the four bodies would then form a central configuration in the four-body problem, except that one of the masses is essentially zero.

Finding the fixed points or *equilibria* in the PCR4BP amount to finding critical points of a special function called the *amended potential.* In practice, these critical points make ideal "parking spaces" for spacecraft, as demonstrated by the Solar and Heliospheric Observatory Satellite and the Wilkinson Microwave Anisotropy Probe located at, or orbiting around, the related Lagrangian points in the planar, circular, restricted three-body problem. Our main result was proving that, for any choice of masses, the number of equilibria in the PCR4BP is finite, bounded above by 196. (The actual number of solutions was just recently proven to be between 8 and 10.) While finiteness may not sound like a significant result, it is actually a difficult and open question as to whether, for a given set of positive masses, the number of central configurations is finite (up to symmetry). This is a famous open problem in celestial mechanics, known as the Smale/Wintner/Chazy question, that has only been solved for the cases $n = 3, 4$, and 5. Thus, our finiteness result fits in well with related work in the field. Our proof incorporated computational algebraic geometry and BKK theory, a theory relating the number of solutions in a system of polynomial equations to the volume of a special geometric figure created from the exponents in the polynomials (a Newton polytope). Chris and Julie were responsible for doing symbolic computations in Maple and numerical calculations in Matlab in order to confirm and establish our results. Our paper, titled "Finiteness in the Planar Restricted Four-Body Problem," appeared in the journal *Qualitative Theory of Dynamical Systems* in 2009 [9].

Linear stability of some symmetric periodic collision orbits

During a week-long visit to the Department of Mathematics at Brigham Young University (BYU) in the fall of 2008, I collaborated with an existing research group to investigate the linear stability of some interesting symmetric periodic orbits in the four-body problem. The team of researchers at BYU consisted of Lennard Bakker, Tiancheng Ouyang, Skyler Simmons (undergraduate), and Duokui Yan (graduate student). The orbits studied were highly symmetric and featured simultaneous binary collisions (two pairs of bodies each undergoing collisions at the same time). Although this sounds strange, it was mathematically possible to continue the solutions past collision, a process known as *regularization*, to produce genuine periodic orbits. Physically speaking, at each collision the planets elastically bounce off each other. Two particular families were discovered and analyzed, a planar and a collinear example.

I was invited to the group because of my previous work studying the linear stability of the famous figure-eight orbit of the three-body problem [4, 15]. A stable periodic solution maintains its shape under perturbation. Stable orbits are important to find because these are the solutions that may actually exist in space (e.g., the Trojan and Greek asteroids). Linear stability is determined by analyzing the effects of first-order perturbations. Mathematically, this means computing the eigenvalues of the monodromy matrix associated to the periodic orbit and checking to see if they lie on the unit circle. An eigenvalue off the unit circle corresponds to a direction in which the periodic solution may be perturbed and lose its form (a direction of instability). My role was to apply the techniques I developed for the figure eight and determine if these new special collision orbits were stable. Due to the symmetry of the orbits, the calculations were similar to those of my previous work.

The stability analysis was conducted on the regularized equation of motion, which, due to symmetry, is a two-degree of freedom Hamiltonian system. Each orbit contains a time-forward and time-reversing symmetry, and can be characterized as the solution to a certain boundary-value problem over a fundamental domain $[0, T]$. Once the solution has been determined over the fundamental domain, the full periodic orbit can be pieced together using the appropriate symmetries. For example, in the planar case, the period is $8T$ and the orbit has a symmetry group isomorphic to D_4, the dihedral group of degree four. Using my symmetry reduction techniques, the linear stability analysis was simplified to the numerical computation of a single real number, a number that depends only on the behavior of the periodic orbit over the fundamental domain.

The advantage of this approach is that it reduces the numerical error inherent in the stability calculation. Since the initial conditions of the orbit are known only numerically, following the orbit to determine its stability leads to numerical inaccuracies. By restricting the size of the domain and the numbers to be computed, a more rigorous result was obtained. Surprisingly, we found linearly stable orbits in both cases. In the collinear case a family of orbits was studied in terms of a mass parameter m and intervals of m-values for which the orbit is linearly stable were located. Based on our joint work, a paper titled "Linear Stability for Some Symmetric Periodic Simultaneous Binary Collision Orbits in the Four-Body Problem" was published in *Celestial Mechanics and Dynamical Astronomy* in 2010 [2].

2. The N-Vortex Problem

Although much of my research has concentrated in celestial mechanics, I have recently begun working in a related area called the n-vortex problem. Although situated in the field of fluid dynamics, an area I am not very familiar with, the special formulation of the planar n-vortex problem is very similar to the set-up used for the n-body problem in celestial mechanics. Many of the techniques I have used in my previous research apply quite nicely in this new setting.

Introduced by Helmholtz in 1858 and later given a Hamiltonian formulation by Kirchhoff in 1876, the n-vortex problem is a widely used model for providing finite-dimensional approximations to vorticity evolution in fluid dynamics. The general goal is to track the motion of the vortices as points rather than focus on their internal structure and deformation, a concept analogous to the use of "point masses" in celestial mechanics. It is important to note that, unlike the n-body problem, where the masses are all assumed positive, in the n-vortex problem, vortices can have either positive or negative circulations depending on their direction of rotation.

Stability of relative equilibria

I have investigated the stability of relative equilibria in the planar n-vortex problem. Recall that a relative equilibrium is a rigidly rotating configuration, a solution to the equations of motion that maintains its shape for all time. In this case, the center of rotation $\mathbf{c}$ is the center of vorticity, a point analogous to the center of mass in the n-body problem, and the vortices are each traveling on circular orbits about $\mathbf{c}$. One of the reasons I was interested in studying this topic is that researchers in fluid dynamics have repeatedly found motion in their simulations that looks remarkably like a relative equilibrium orbit. For instance, scientists trying to model the intensity in the eyewalls of hurricanes have found rotating structures reminiscent of these special types of solutions. Given their persistence in models of physical phenomena, analyzing the stability of relative equilibria in the planar n-vortex problem is of great interest and may have some practical significance.

In 2013 I published the paper "Stability of Relative Equilibria in the Planar n-Vortex Problem" in the *SIAM Journal on Applied Dynamical Systems* [17]. (SIAM stands for the Society for Industrial and Applied Mathematics.) The main result of this paper was the following classification theorem for linearly stable relative equilibria: For the case of positive circulations, a relative equilibrium is linearly stable if and only if it is a nondegenerate minimum of the Hamiltonian restricted to a level surface of the angular impulse. This is a strong and concise result that completely classifies the stable solutions in terms of a topological characteristic. Some deep implications follow relatively easily. First, using a criterion of Dirichlet's, I proved that any linearly stable relative equilibrium with positive vorticities is also nonlinearly stable. In essence, a solution that starts near a stable relative equilibrium orbit will remain close to that orbit (as a set) for all forwards and backwards time. Second, I showed that for a generic choice of positive circulations, there exists a non-collinear stable relative equilibrium orbit. This helps explain the prevalence and duration of relative equilibrium type motions in the numerical simulations of the eyewall of hurricanes and in related experiments in fluid dynamics. The existence result for the vortex case is in stark contrast to the n-body problem, where I showed in my doctoral thesis that for n sufficiently large, there are *no* linearly stable relative equilibria with equal masses [16].

In addition to the above results, I also carefully analyzed the linear stability of two symmetric families of relative equilibria, the rhombus and the isosceles trapezoid, with stable solutions found in each case. In each family, the four vortices are split into two equal-strength pairs. For the rhombus case, there are two geometrically distinct families, one of which is stable even when the pairs of vortices have circulations of opposite signs. The stability type for this particular family changes at an important value of the circulation, where the total angular vortex momentum vanishes. The other rhombus solution is always unstable. However, a pitchfork bifurcation occurs in this family, and I was able to show rigorously the corresponding change in eigenvalues from a real pair to a pure imaginary pair.

Relative equilibria with two pairs of equal vorticities

Another topic that I have studied in the n-vortex problem is the classification of all relative equilibria having four vortices, where two pairs of vortices are assumed to have equal strength. I worked on this problem with two collaborators, Marshall Hampton (University of Minnesota Duluth) and Manuele Santoprete (Wilfrid Laurier University, Waterloo, Ontario). Specifically, we assume that one pair of vortices each has a circulation of one, while the vortices in the other pair each have a strength of m, treating m as a real parameter (both positive and negative values are allowed.) Our lengthy paper (54 pages), titled "Relative Equilibria in the Four-Vortex Problem with Two Pairs of Equal Vorticities," was published in the *Journal of Nonlinear Science* in 2014 [8].

At the start of our research, we had no idea how far we would get working on this problem. Our initial intent was to explore the shape and symmetry of the solutions and how this depends on the parameter m. One of our main results was proving that when $m > 0$, the convex configurations (those with no interior vortex) *must* contain a line of symmetry, forming a rhombus or an isosceles trapezoid. If $m < 0$, it is the concave solutions that must contain a line of symmetry. In this case, the solution is an isosceles triangle with an interior vortex on the axis of symmetry.

These results answer a question whose counterpart in celestial mechanics remains unsolved. It is an open question as to whether or not a four-body convex central configuration with two pairs of equal masses must have a line of symmetry. If the equal pairs of masses are situated across from each other, than the configuration must be a rhombus; however, if the equal masses are adjacent, then symmetry has not been shown to follow necessarily. We were excited to discover that it was possible to solve this problem in the vortex setting.

As we made progress in our work, it became apparent that we could go deeper, providing a *complete* classification of all types of four-vortex relative equilibria for these particular choice of circulations (see Table 1 in [8]). In essence, once we knew the shape of the configuration for a range of m-values, it was possible to count the number of solutions for that particular shape. In some special cases, such as the rhombus and isosceles trapezoid families, we derived specific formulas for the positions of each vortex in terms of m. Our techniques are rigorous, involving a combination of analysis, and modern and computational algebraic geometry.

We also investigated some interesting bifurcations that occur as the parameter m is varied. For instance, a pitchfork bifurcation takes place when m is approximately -0.5951 (the precise value is determined by the root of a cubic polynomial). As m increases through this special value the rhombus solution bifurcates into two convex kite configurations, each having the two positive strength vortices on the axis of symmetry. The rhombus solution becomes degenerate at the bifurcation, but persists—hence the pitchfork bifurcation. Interestingly, my later research on the linear stability of relative equilibria [17] confirmed the existence of this bifurcation.

My work in the n-vortex problem was recently highlighted in a special issue of *International Innovation*, a publication designed to promote and disseminate scientific research [12]. The title of this particular issue is "What is Mathematics?"

3. Numerical Methods and Complex Dynamics

One of the most common iterative algorithms for finding solutions to an equation is *Newton's method*. Given an equation $f(x) = 0$ and an initial guess x_0, Newton's method provides a better guess given by

$$x_1 \;=\; N_f(x_0) \;=\; x_0 - \frac{f(x_0)}{f'(x_0)}\,.$$

Here, the numerical technique uses information about the first derivative of f at x_0 to obtain an improved approximation x_1. The process repeats, generating a sequence of numbers $x_0, x_1, x_2, \ldots$ that hopefully converges to a solution of the equation. More accurate numerical methods, such as *Halley's method*, utilize information from higher-order derivatives of f.

Interestingly, numerical methods can fail quite drastically. For example, applying Newton's method to the equation $x^2 + 1 = 0$ with a real initial guess x_0 will *never* converge to a solution because the solutions are complex. Even worse, applying Newton's method to find the roots of $p(z) = z^4 - 6z^2 - 11$ leads to an entire region of the complex plane for which initial seeds eventually bounce back and forth between 1 and -1, neither of which are solutions to $p(z) = 0$. In this case, because $z = 1$ and $z = -1$ are critical points of N_p that attract nearby seeds, they are said to lie on a *superattracting* cycle of period two. It is this failure of the numerical method to converge on an open set of initial guesses that I find appealing.

I have studied this problem from a complex dynamical systems perspective. Given a complex polynomial $p(z)$, its roots are superattracting fixed points of the map determined by applying the numerical method to p. For the method to break down on an open set, there must be another attracting periodic cycle that draws in entire regions of the complex plane. An important theorem from complex dynamics, due to Julia and Fatou in the 1920's, states that the basin of an attracting cycle must contain a critical point of the map [3]. Therefore, by following the orbits of the "free" critical points, those other than the roots themselves, we can determine if an extraneous attracting cycle exists. The free critical points for Newton's method are the inflection points of p.

One fruitful approach I have used is to study a given numerical method applied to a particular family of polynomials. In the paper "Elusive Zeros Under Newton's Method," Trevor O'Brien (HC '05) and I studied Newton's method on the quartic family

$$q_\lambda(z) \;=\; (z-1)(z+1)(z-\lambda)(z-\bar{\lambda}),$$

where $\lambda \in \mathbb{C}$ is a complex parameter. The family q_λ was chosen for the symmetric location of its roots, but also because it has two free critical points to follow dynamically. The problem turned out to have many similarities with Milnor's work on iterating the general complex cubic [11], in part because each problem has two free critical points. Of particular interest was the discovery of figures in the parameter plane reminiscent of both quadratic phenomena (Mandelbrot-like sets and tricorns) as well as cubic behavior (swallow and product configurations). This discrepancy occurs because q_λ'' has real coefficients and therefore the two free critical points are either real or complex conjugates. In the real case, there are two critical points to follow whose orbits are unrelated. In contrast, the orbits of a pair of complex conjugate critical points will be conjugate, so there is essentially only one critical point to examine, leading to quadratic-like behavior.

It is possible to explain much of the complicated parameter plane picture by following the orbits of the critical points of a real one-dimensional map. The large amount of symmetry present in our family leads to this nice reduction. Applying both numerical and analytic methods, various phenomena in the parameter plane were predicted. Using a bisection algorithm, we numerically located an abundance of λ-values along the imaginary axis for which Newton's method applied to q_λ has a superattracting periodic cycle. Applying Milnor's insights, the period and type of attracting cycle found governs much of the structure in the parameter plane along the imaginary axis. For the case where λ is real, we rigorously proved that both free critical points converge to roots of the polynomial.

Trevor conducted research on this problem as part of his senior thesis. We initially hoped to showcase our work as a chapter in a special book on chaos and fractals to be published by the Mathematical Association of America. However, our paper was rejected, being deemed too technical for the intended audience. After putting the work aside for several years, I revisited the manuscript in the spring of 2014, transforming and updating the exposition into a more suitable form for a research journal. The paper was then published in the journal *Applied Mathematics* in 2014, and to our great satisfaction, one of Trevor's figures was chosen for the cover of the journal [13].

4. Research with Undergraduates

One important feature of my research is that I have been able to find problems that are accessible to talented and motivated undergraduates. To date, I have mentored and collaborated with 15 undergraduate researchers and published five papers with undergraduate co-authors. The names, topics, and accomplishments of the students I have worked with since receiving tenure can be found in the *Research with Undergraduates* section of my portfolio.

One of the ways that the National Science Foundation (NSF) recognizes the importance of undergraduate research is through their Research at Undergraduate Institutions (RUI) grants. I have been fortunate to receive two NSF-RUI grants, and it is likely that part of my success in obtaining these grants stems from an interest in, and plan for, incorporating undergraduates into my research program. Funds from these grants have been used to support 10 undergraduate researchers, providing resources such as a summer stipend for the student, payment of room and board during the Holy Cross Summer Research Program, money to purchase books and software, and reimbursement of travel expenses incurred by students who present their work at conferences.

Based on my experience working with undergraduate researchers and inspired by some ideas generated at the Conference on Innovation in Undergraduate Teaching and Research (Montclair State University, June 4-5, 2008), I wrote the paper "Conducting Mathematical Research with Undergraduates," which was published in the journal *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies* in 2013 [14]. The article describes some of the key issues in undergraduate mathematical research: selecting good research students, finding appropriate research questions, mentoring versus collaboration, and presenting and publishing student work. Some useful professional and financial resources supporting undergraduate research are also highlighted in the paper.

I have found my student researchers to be enthusiastic young scholars who are willing to tackle difficult problems. Many of my students have demonstrated a great deal of creativity, curiosity, and perseverance in their research. Several recent students have produced some noteworthy results, results that I expect will lead to publication once I find the time to organize and compile their work into a suitable scholarly format. I look forward to continuing my collaboration with undergraduates as an important facet of my research program.

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