Cyclic Central Configurations in the Four-Body Problem

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Definition

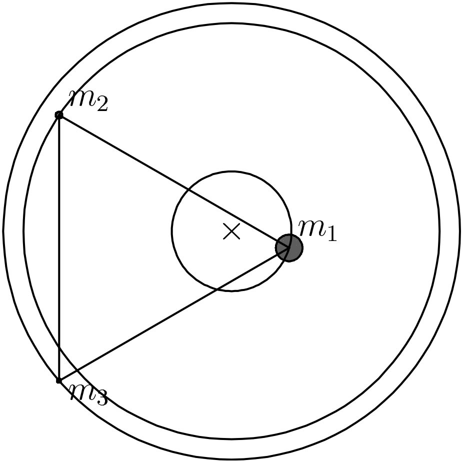
A planar *central configuration* (c.c.) is a configuration of bodies $(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n), \mathbf{x}_i \in \mathbb{R}^2$ such that the acceleration vector for each body is a common scalar multiple of its position vector (with respect to the center of mass). Specifically, in the Newtonian *n*-body problem with center of mass $\mathbf{c}$, for each index i,

$$\sum_{j \neq i}^{n} \frac{m_i m_j (\mathbf{x}_j - \mathbf{x}_i)}{||\mathbf{x}_j - \mathbf{x}_i||^3} + \lambda\, m_i (\mathbf{x}_i - \mathbf{c}) = 0$$

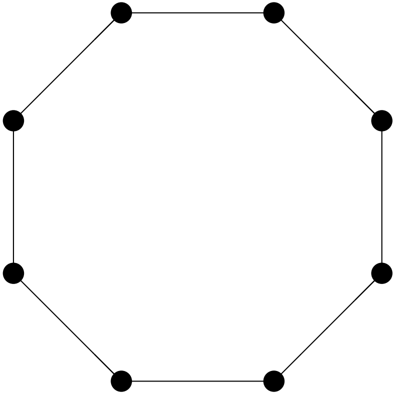
for some scalar λ.

- Finding c.c.'s is an algebra problem — no dynamics or derivatives.
- Summing together the n equations above quickly yields $\mathbf{c} = \frac{1}{M} \sum m_i \mathbf{x}_i$.

Equilateral Triangle (Lagrange 1772)



Regular n-gon (equal mass required for $n \geq 4$)



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- If the center of mass coincides with the center of the circle, the only possibility is the square with equal masses (Hampton, 2003). If such a co-circular c.c. existed for more bodies, other than the regular n-gon, this would be a choreography (all bodies tracing out the same curve) **without** equal time spacing between bodies.

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- Spatial 5-body pyramidal c.c.'s exist where four bodies lie on a sphere with the fifth body at the center of the sphere. The four bodies forming the base of the configuration are co-circular and form a c.c.c. in the 4-body problem (Fayçal, 1996; Albouy, 2003)

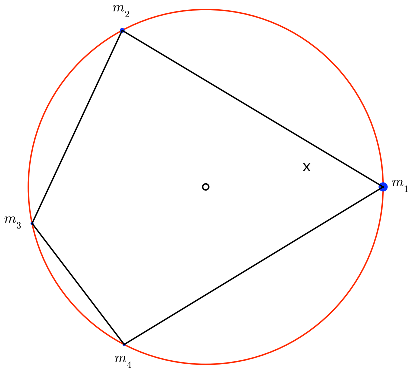

Figure: An example of a cyclic central configuration. The center of the circumscribing circle is marked with an O while the center of mass is labeled with an X.

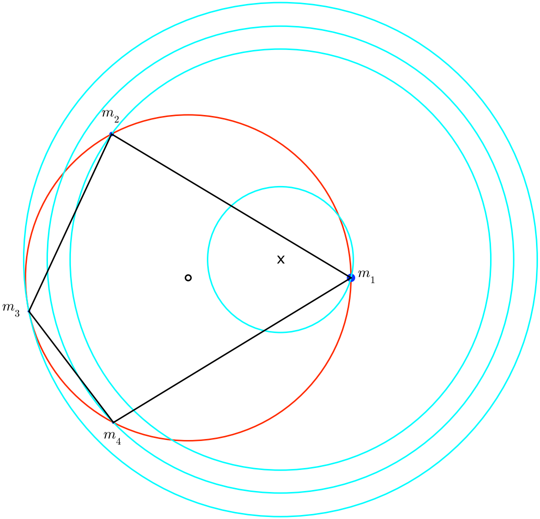

Figure: The relative equilibrium generated by the previous c.c.c.

Mutual Distances Make Great Coordinates

Newtonian potential function:

$$U(\mathbf{q}) = \sum_{i<j}^{4} \frac{m_i m_j}{r_{ij}}$$

Moment of Inertia:

$$I(\mathbf{q}) = \frac{1}{2M} \sum_{i<j}^{4} m_i m_j r_{ij}^2$$

where $M = m_1 + \cdots + m_4$ is the total mass.

Problem: The six variables $r_{12}, r_{13}, r_{14}, r_{23}, r_{24}$ and r_{34} are not independent in the planar problem. Generically, they describe a tetrahedron, not a planar configuration. It is easy to see that the regular tetrahedron is the only non-planar c.c. in the four-body problem.

The Cayley-Menger Determinant

To use the six mutual distances $r_{12}, r_{13}, r_{14}, r_{23}, r_{24}, r_{34}$ as variables, we need an additional constraint that ensures the configuration is planar. We require that the volume of the tetrahedron formed by the four bodies be zero (Cayley-Menger determinant).

$$V = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & r_{12}^2 & r_{13}^2 & r_{14}^2 \\ 1 & r_{12}^2 & 0 & r_{23}^2 & r_{24}^2 \\ 1 & r_{13}^2 & r_{23}^2 & 0 & r_{34}^2 \\ 1 & r_{14}^2 & r_{24}^2 & r_{34}^2 & 0 \end{vmatrix} = 0$$

Using Ptolemy's Theorem

If four bodies lie on a common circle and are numbered sequentially (ie. the diagonals have lengths r_{13} and r_{24}), then $P = 0$, where

$$P = r_{12}r_{34} + r_{14}r_{23} - r_{13}r_{24}.$$

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Theorem (Apostol, 1967): For any convex quadrilateral numbered sequentially or for any tetrahedron, $P \geq 0$ with equality iff the four bodies lie on a circle.

Let $\mathcal{P} \subset \mathcal{G}$ denote the set of all geometrically realizable $\mathbf{r}$ satisfying $P(\mathbf{r}) = 0$. Then, $\mathbf{r}$ is an element of $\mathcal{P}$ iff it corresponds to a cyclic quadrilateral with a sequential ordering. Any element of $\mathcal{P}$ satisfies both $V(\mathbf{r}) = 0$ and $P(\mathbf{r}) = 0$.

Avoiding the Cayley-Menger Determinant

Note: Apostol's result shows that the two co-dimension one level surfaces $\{V = 0\}$ and $\{P = 0\}$ (in $\mathbb{R}^{+^6}$) meet tangentially at any point in $\mathcal{P}$.

Lemma

For any $\mathbf{r} \in \mathcal{P}$,

$$\nabla V(\mathbf{r}) \;=\; \left(\frac{4}{r_c^2} \prod_{i<j} r_{ij} \right) \nabla P(\mathbf{r})$$

where r_c *is the circumradius of the cyclic quadrilateral. In other words, on the the set of geometrically realizable vectors for which both* V *and* P *vanish, the gradients of these two functions are parallel.*

How to find C.C.C.'s

This calculation uses the remarkable relation

$$\frac{\partial V}{\partial r_{ij}^2} = -32\,\Delta_i \Delta_j,$$

where Δ_i is the oriented area of the triangle containing all bodies except for the i-th body. Also, since the bodies lie on a common circle, $|\Delta_i| = r_{jk} r_{kl} r_{jl}/(4r_c)$.

Corollary

A four-body cyclic central configuration $\mathbf{r}$ *is a critical point of the function*

$$U + \lambda M(I - I_0) + \sigma P$$

satisfying $I = I_0, P = 0$ *and* $V = 0$.

Dziobek's Equations

Using the six mutual distances as variables, we find

$$m_1 m_2(r_{12}^{-3} - \lambda) = \sigma \frac{r_{34}}{r_{12}}, \qquad m_3 m_4(r_{34}^{-3} - \lambda) = \sigma \frac{r_{12}}{r_{34}}$$

$$m_1 m_3(r_{13}^{-3} - \lambda) = -\sigma \frac{r_{24}}{r_{13}}, \qquad m_2 m_4(r_{24}^{-3} - \lambda) = -\sigma \frac{r_{13}}{r_{24}}$$

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\end{aligned}
$$

This yields a well-known relation of Dziobek (1900)

$$
(r_{12}^{-3} - \lambda)(r_{34}^{-3} - \lambda) = (r_{13}^{-3} - \lambda)(r_{24}^{-3} - \lambda) = (r_{14}^{-3} - \lambda)(r_{23}^{-3} - \lambda) \tag{1}
$$

which must be true for any planar 4-body c.c., not just cyclic c.c.'s.

Eliminating λ from equation (1) in a clever way yields

$$(r_{13}^3 - r_{12}^3)(r_{23}^3 - r_{34}^3)(r_{24}^3 - r_{14}^3) \;=\; (r_{12}^3 - r_{14}^3)(r_{24}^3 - r_{34}^3)(r_{13}^3 - r_{23}^3). \quad (2)$$

Equation (2) is necessary and sufficient for a 4-body planar c.c. given that the six mutual distances determine a geometrically realizable planar configuration. However, it does *not* ensure positivity of the masses.

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Generically, the space of 4-body planar c.c.'s is three dimensional since it is described by the equations $V = 0$, $I = I_0$ and equation (2). Restricting to a circle yields a two dimensional space.

Mass Ratios

$$\frac{m_2}{m_1} = \frac{(\lambda - r_{13}^{-3})\, r_{13} r_{14}}{(r_{23}^{-3} - \lambda)\, r_{23} r_{24}} = \frac{(r_{14}^{-3} - \lambda)\, r_{13} r_{14}}{(\lambda - r_{24}^{-3})\, r_{23} r_{24}}$$

$$\frac{m_3}{m_1} = \frac{(r_{12}^{-3} - \lambda)\, r_{12} r_{14}}{(r_{23}^{-3} - \lambda)\, r_{23} r_{34}} = \frac{(r_{14}^{-3} - \lambda)\, r_{12} r_{14}}{(r_{34}^{-3} - \lambda)\, r_{23} r_{34}}$$

$$\frac{m_4}{m_1} = \frac{(r_{12}^{-3} - \lambda)\, r_{12} r_{13}}{(\lambda - r_{24}^{-3})\, r_{24} r_{34}} = \frac{(\lambda - r_{13}^{-3})\, r_{12} r_{13}}{(r_{34}^{-3} - \lambda)\, r_{24} r_{34}}.$$

Without loss of generality, let $r_{12} = 1$ be the longest exterior side and let $r_{14} \geq r_{23}$. Requiring positive masses gives

$$r_{13} \geq r_{24} > r_{12} = 1 \geq r_{14} \geq r_{23} \geq r_{34}.$$

The diagonals are longer than any of the exterior sides. The longest exterior side is opposite the smallest.

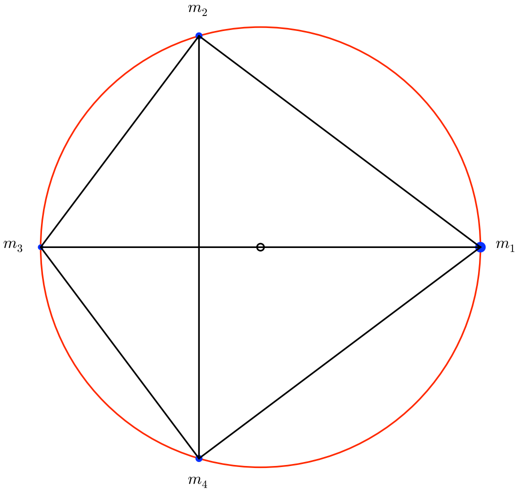

Figure: An example of a cyclic kite central configuration with $m_2 = m_4$. The center of the circumscribing circle is marked with an O.

Symmetric Example I: Kite Configurations

Theorem

There exists a one-parameter family of cyclic kite central configurations with bodies one and three lying on the diameter of the circumscribing circle. The masses are $m_1 = 1, m_2 = m_4 = m$ and $m_3 = \alpha\, m$ and are ordered $m_1 \geq m_2 = m_4 \geq m_3$ with equality iff the configuration is a square. At one end of the family ($x = 1/\sqrt{3}$) is a c.c. of the planar, restricted 4-body problem, with bodies 1, 2 and 4 forming an equilateral triangle and $m_3 = 0$. At the other end ($x = 1$) is the square with equal masses.

$$m = \frac{4x(c^3 - 1)}{c(8 - c^3)}, \quad \alpha = \frac{c(8x^3 - c^3)}{4(c^3 - x^3)}, \quad c = \sqrt{1 + x^2}$$

To ensure positive masses, we must have $1/\sqrt{3} < x \leq 1$.

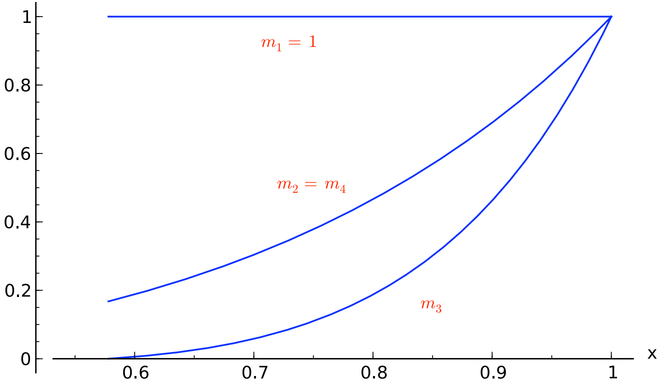

Figure: The values of the masses for the cyclic kite c.c.'s

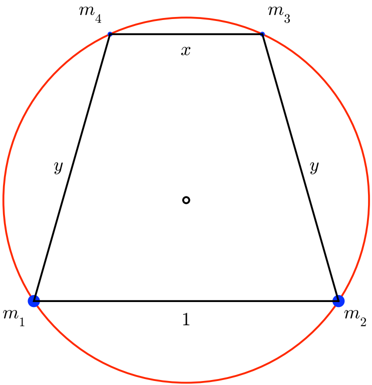

Figure: An example of an isosceles trapezoid central configuration. The center of the circumscribing circle is marked with an O.

Symmetric Example II: Isosceles Trapezoid

Let $x = r_{34}$ and $y = r_{14} = r_{23}$. To be a c.c., equation (2) must be satisfied. This yields the constraint

$$T \;=\; (y^2 + x)^{3/2}(2y^3 - x^3 - 1) - y^3 - x^3y^3 + 2x^3 \;=\; 0.$$

Theorem

For each value of $x \in (0, 1]$, there exists a unique value of $y \in [x, 1]$ such that $T(x, y) = 0$. Moreover, the distance parameter y can be written as a differentiable function of x.

Examining $T = 0$, we see that while the smallest side of the trapezoid (parallel to the base) can range from 0 to 1, the length of the congruent legs is considerably constrained between approximately 0.908 and 1.

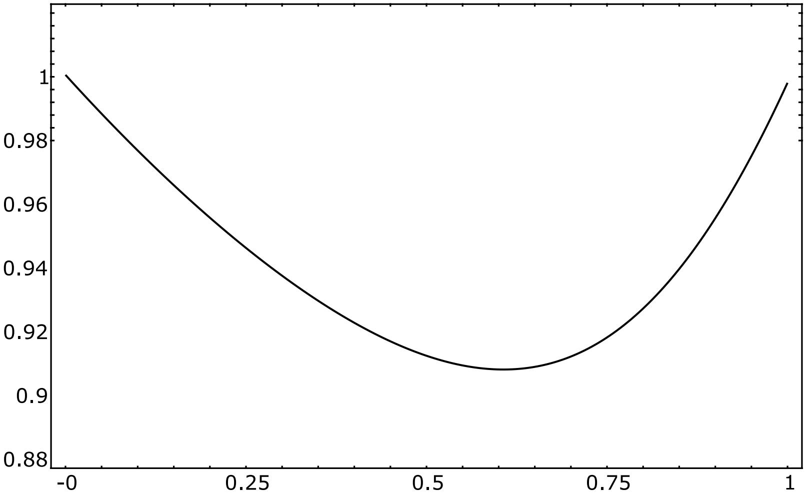

Figure: The relationship between the two distances $x = r_{34}$ and $y = r_{14} = r_{23}$ in the isosceles trapezoid family of c.c.'s.

$$\Omega \;=\; \{\mathbf{r} \in \mathbb{R}^{+^6} : r_{13} \;\geq\; r_{24} \;>\; r_{12} = 1 \;\geq\; r_{14} \;\geq r_{23} \;\geq\; r_{34}\}.$$

For a cyclic quadrilateral ordered sequentially, the lengths of each diagonal r_{13} and r_{24}, with $r_{12} = 1$, can be written as

$$r_{13} \;=\; \left(\frac{r_{14}^2 r_{23} + r_{14} r_{34}(r_{23}^2 + 1) + r_{23} r_{34}^2}{r_{23} + r_{14} r_{34}} \right)^{1/2} \tag{3}$$

$$r_{24} \;=\; \left(\frac{r_{14}^2 r_{23} r_{34} + r_{14}(r_{34}^2 + r_{23}^2) + r_{23} r_{34}}{r_{14} + r_{23} r_{34}} \right)^{1/2} . \tag{4}$$

Eliminating the Diagonals

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If these equations hold, then the configuration is both planar and cyclic. Substituting (3), (4) and $r_{12} = 1$ into

$$(r_{13}^3 - r_{12}^3)(r_{23}^3 - r_{34}^3)(r_{24}^3 - r_{14}^3) - (r_{12}^3 - r_{14}^3)(r_{24}^3 - r_{34}^3)(r_{13}^3 - r_{23}^3) = 0$$

yields a complicated equation in three variables,

$$F(r_{14}, r_{23}, r_{34}) = 0.$$

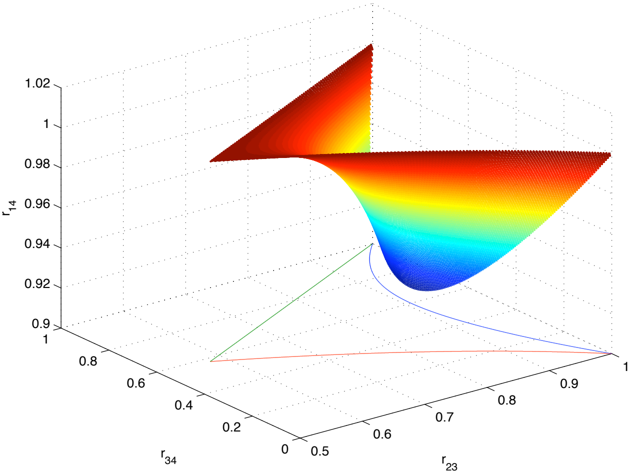

Figure: The surface Γ of cyclic central configurations in $r_{34}r_{23}r_{14}$-space. The outline of the projection onto the $r_{34}r_{23}$-plane is shown plotted in the plane $r_{14} = 0.9$. Figure generated with Matlab using a bisection algorithm.

The Surface of C.C.C.'s is a Graph

Theorem

The set of cyclic central configurations Γ *is the graph of a differentiable function* $r_{14} = f(r_{34}, r_{23})$ *over the two exterior side-lengths* r_{34} *and* r_{23}*. The domain of this function is the region*

$$\mathcal{D} = \{(r_{34}, r_{23}) \in \mathbb{R}^{+^2} : 1 \geq r_{23} \geq r_{34}, r_{23} \leq \tau(r_{34}), r_{23}^2 + r_{34}^2 + r_{34}r_{23} > 1\}$$

where τ *is defined implicitly by* $T(x, \tau(x)) = 0$*.*

Boundaries of $\mathcal{D}$:

Kite: $r_{23} = r_{34}$

Trapezoid: $r_{23} = \tau(r_{34})$

PCR4BP (Equilateral Triangle, $m_3 = 0$): $r_{23}^2 + r_{34}^2 + r_{34}r_{23} = 1$

Ordering the Masses

Theorem

Any cyclic central configuration in Γ satisfies

$$1 = m_1 \;\geq\; m_2 \;\geq\; m_4 \;\geq\; m_3.$$

In other words, the largest body is located at the vertex between the two longest exterior sides, and the smallest body is opposite the largest one. In addition, the two largest bodies lie on the longest side while the two smallest bodies lie on the smallest side.

Proof relies on the formula for the mass ratios and estimates involving the mutual distances arising from the fact that the four bodies lie on a common circle.

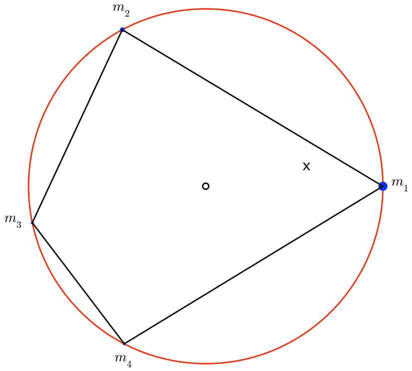

Figure: Ordering the masses: $1 = m_1 \ \geq \ m_2 \ \geq \ m_4 \ \geq \ m_3$.

Symmetry and Masses

Corollary

If just two bodies of a cyclic central configuration have equal mass, then the configuration is symmetric, either a kite or an isosceles trapezoid. Specifically, for any cyclic c.c. in Γ, if either $m_1 = m_2$ or $m_3 = m_4$, then the configuration is an isosceles trapezoid (and the other pair of masses must be equal). If $m_2 = m_4$, the configuration is a kite. If any three masses are equal, the configuration must be a square.

Some Remarks Concerning the Masses

1. For certain choices of positive masses, **no** cyclic c.c. exists. Generically choosing a mass vector of the form $(m_1, m_2 = m_1, *, *)$ will not yield a cyclic central configuration since $m_3 = m_4$ is required to make it an isosceles trapezoid or a specific relationship between m_3 and m_4 is required to make the configuration a kite.

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2. In the 4-body problem, for a given choice and ordering of the masses, there exists a convex central configuration – MacMillan and Bartky (1932), Xia (2004). It is not known whether this configuration is unique. If we restrict to cyclic c.c.'s, we *conjecture* that the configuration is indeed unique (numerical).

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4. While m_3 and m_4 can each range from 0 to 1 (e.g., in the trapezoid family), it appears (numerical) that m_2 is bounded below by $(8 - 3\sqrt{3})/(12\sqrt{3} - 4) \approx 0.167$.

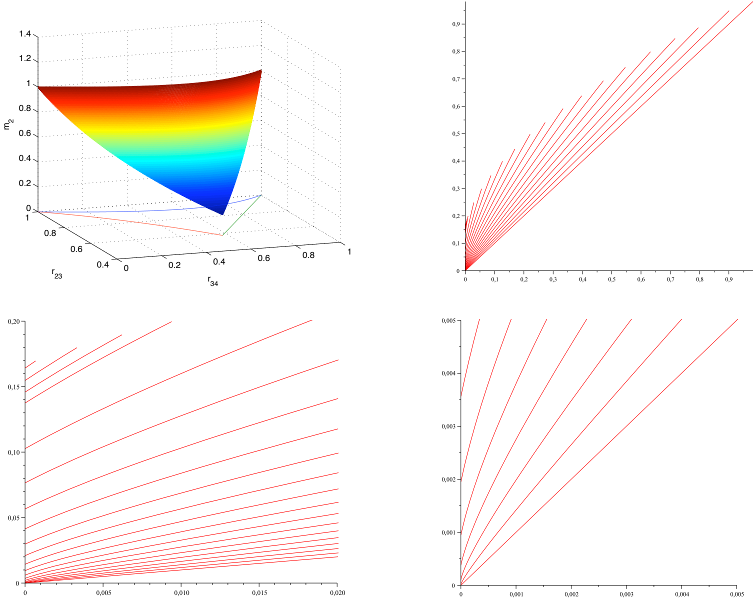

Figure: Some numerical evidence for uniqueness.

Future Work/Ideas:

1. Let θ_{12} be the arc along the circumscribing circle between bodies 1 and 2. Using the Perpendicular Bisector Theorem and some simple geometry, an upper bound for θ_{12} is 144°. However, numerical calculations show that it is most likely 120° and that θ_{12} decreases as r_{23} increases through $\mathcal{D}$. Prove this analytically?

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3. Linear stability of the corresponding relative equilibria?

4. Extension to general four-body convex c.c.'s. Instead of $P = 0$, is constraining to $P = c$ useful? Classification? Uniqueness? Are there other useful constraints tangent to $V = 0$?