Classifying Four-Body Convex Central Configurations

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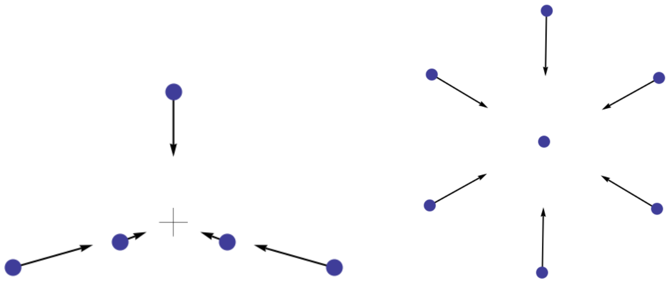
Dynamical Systems Seminar
Boston University
October 1, 2018

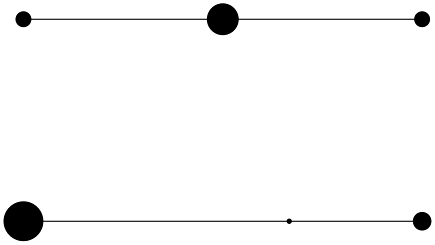
Central Configurations



The gravitational force on each body points toward the center of mass and is proportional to the distance from the center of mass.

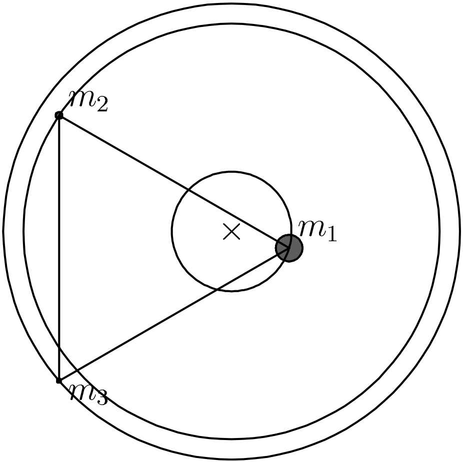
Figures by Rick Moeckel (2014), Scholarpedia, 9(4):10667.

3-Body Collinear Configurations (Euler 1767)



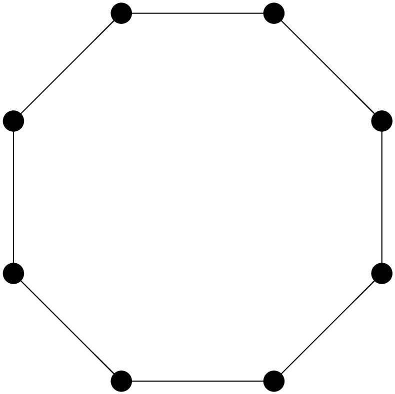
For each ordering of n arbitrary masses on a line, there exists a unique central configuration (Moulton, 1910).

Equilateral Triangle (Lagrange 1772)

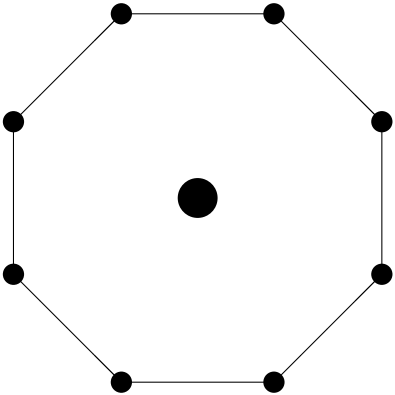


The equilateral triangle is a c.c. for any choice of masses.

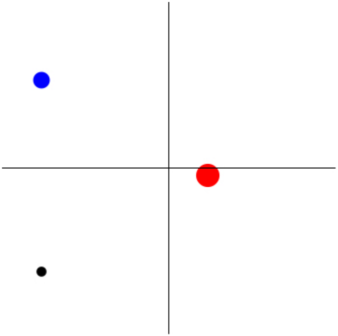
Regular n-gon (equal mass required for $n \geq 4$)



$1 + n$-gon (arbitrary central mass)

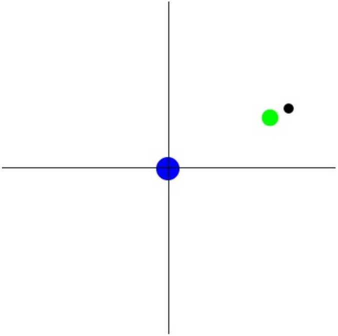


Used by Sir James Clerk Maxwell in 1859 in Stability of the Motion of Saturn's Rings (winner of the Adams Prize).

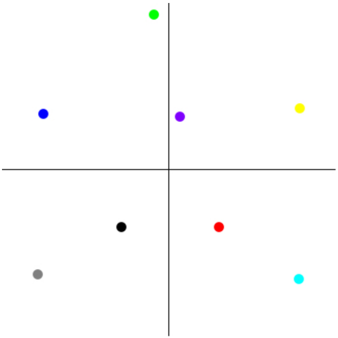


Released from rest, a central configuration maintains the same shape as it heads toward total collision (homothetic motion).

Simulation by Rick Moeckel (2014), Scholarpedia, 9(4):10667.



Given the correct initial velocities, a central configuration will rigidly rotate about its center of mass. Such a solution is called a relative equilibrium.



Any Kepler orbit (elliptic, hyperbolic, ejection-collision) can be attached to a central configuration to obtain a solution to the full *n*-body problem. Above is an example of an asymmetric 8-body c.c. with elliptic homographic motion (eccentricity 0.8).

Simulation by Rick Moeckel (2014), Scholarpedia, 9(4):10667.

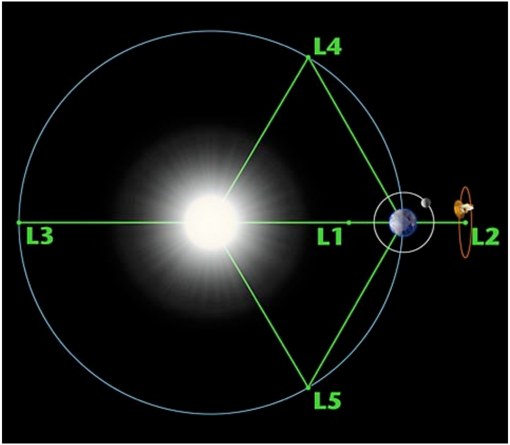


Figure: The five libration points (Lagrange points) in the Sun-Earth system (not drawn to scale). In general, L_4 and L_5 are linearly stable provided the ratio m_{sun}/m_p is sufficiently large. These make great "parking spaces." In February 2017, the OSIRIS-REX mission spent 10 days looking for asteroids at Earth's L_4 point.

Source: http://map.gsfc.nasa.gov/mission/observatory_l2.html

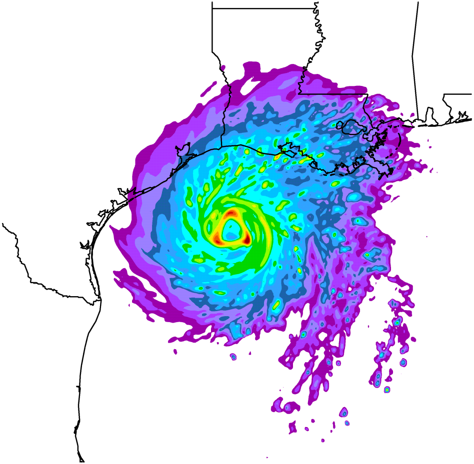


Figure: Weather research and forecasting model from the National Center for Atmospheric Research (NCAR) showing the field of precipitable water for Hurricane Rita (2005). Note the presence of three maxima near the vertices of an equilateral triangle contained within the hurricane's “polygonal” eyewall.
http://www.atmos.albany.edu/facstaff/kristen/wrf/wrf.html

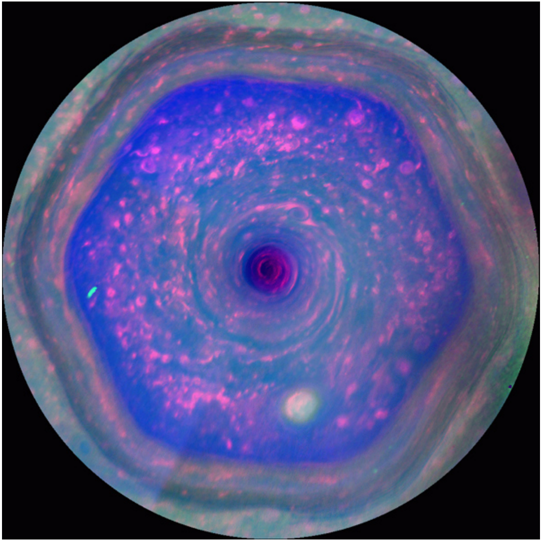


Figure: Saturn's North Pole and its encircling hexagonal cloud structure. First photographed by Voyager in the 1980's and here again recently by the Cassini spacecraft – a remarkably stable structure!

Definition

A planar *central configuration* (c.c.) is a configuration of bodies $x = (x_1, x_2, \ldots, x_n), x_i \in \mathbb{R}^2$ such that the acceleration vector for each body is a common scalar multiple of its position vector (with respect to the center of mass). Specifically, in the Newtonian n-body problem with center of mass c, for each index i, $\frac{\partial U}{\partial q_i}(x) = -\lambda m_i(x_i - c)$ or

$$\sum_{j \neq i}^{n} \frac{m_i m_j (x_j - x_i)}{r_{ij}^3} + \lambda\, m_i(x_i - c) = 0$$

for some scalar λ independent of i, where $r_{ij} = ||x_j - x_i||^3$.

- $U(q) = \sum_{i<j}^{n} \frac{m_i m_j}{r_{ij}}$ is the Newtonian potential function.
- Finding c.c.'s is an algebra problem — no dynamics or derivatives.
- Summing together the n equations above quickly yields $c = \frac{1}{M} \sum m_i x_i$.

Properties of Central Configurations

- Released from rest, a c.c. maintains the same shape as it heads toward total collision (homothetic motion).
- Given the correct initial velocities, a c.c. will rigidly rotate about its center of mass. Such a solution is called a relative equilibrium.
- Any Kepler orbit (elliptic, hyperbolic, ejection-collision) can be attached to a c.c. to obtain a solution to the full n-body problem.
- For any collision orbit in the n-body problem, the colliding bodies asymptotically approach a c.c.
- Bifurcations in the topology of the integral manifolds in the planar problem (holding hc^2 constant where h is the value of the energy and c is the length of the angular momentum vector) occur precisely at values corresponding to central configurations.
- 320 articles found on MathSciNet using a general search for "central configurations" and MSC 70F10

Symmetries

Suppose that $x \in \mathbb{R}^{2n}$ is a central configuration. The following are also central configurations:

1. $kx = (kx_1, \ldots, kx_n)$ for any $k > 0$ (scaling; $c \mapsto kc, \lambda \mapsto \lambda/k^3$)
2. $x - s = (x_1 - s, \ldots, x_n - s)$ for any $s \in \mathbb{R}^2$ (translation; $c \mapsto c - s$)
3. $Ax = (Ax_1, \ldots, Ax_n)$ where $A \in SO(2)$ (rotation; $c \mapsto Ac$)

Thus, central configurations are not isolated. It is standard practice to fix a scaling and center of mass c, and then identify solutions that are equivalent under a rotation.

Note: Reflections of x are also central configurations (e.g., multiplying the first coordinate of c and each x_i by -1), but these are regarded as distinct solutions.

An Alternate Characterization of CC's

The system of equations defining a central configuration can be written more compactly as

$$\nabla U(x) + \lambda \nabla I(x) \; = \; 0, \qquad (1)$$

where I is one half the moment of inertia, $I = \frac{1}{2}\sum_{i=1}^{n} m_i ||q_i - c||^2$.

Thus, a central configuration is a critical point of U subject to the constraint $I = k$ (the mass ellipsoid). This gives a useful topological approach to studying central configurations (Smale, Conley, Meyer, McCord, Moeckel, Ferrario, etc.)

Smale/Wintner/Chazy Question: For a fixed choice of masses, is the number of equivalence classes of planar central configurations finite? (Smale's 6th problem for the 21st century)

Research Goals

1. Describe the set of all convex four-body central configurations. Convex means that no body is contained within the convex hull of the other three. Dimension? Coordinates? Boundaries?

2. Classify the convex central configurations according to symmetry or a special geometric property (kite, trapezoid, rhombus, co-circular, equidiagonal, tangential, etc.). What is the dimension of each subset and how is it situated within the larger space?

3. Show that there is a unique central configuration for any fixed choice of positive masses in a prescribed order. This is Problem #10 on a published list of open problems in celestial mechanics by Albouy, Cabral, and Santos (2012). Existence was proven by MacMillan and Bartky (1932).

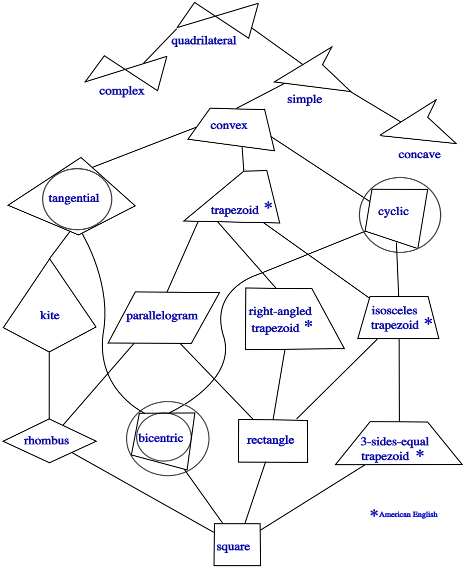


Figure: Classifying convex quadrilaterals. Which of the above can be a central configuration? Dimension of each subset? Image Source: Jlipskoch Alexgabi

Mutual Distances

Treating the six mutual distances $r_{12}, r_{13}, r_{14}, r_{23}, r_{24}, r_{34}$ as variables, a central configuration is a critical point of

$$U - \lambda(I - I_0) - \frac{\mu}{32} V$$

subject to the constraints $I = I_0$ and $V = 0$, where $I = \frac{1}{2M} \sum_{i<j} m_i m_j r_{ij}^2$, and V is the Cayley-Menger determinant

$$V = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & r_{12}^2 & r_{13}^2 & r_{14}^2 \\ 1 & r_{12}^2 & 0 & r_{23}^2 & r_{24}^2 \\ 1 & r_{13}^2 & r_{23}^2 & 0 & r_{34}^2 \\ 1 & r_{14}^2 & r_{24}^2 & r_{34}^2 & 0 \end{vmatrix}.$$

Key Formula: $\frac{\partial V}{\partial r_{ij}^2} = -32\, A_i A_j$ where A_i is the signed area of the triangle whose vertices contain all bodies except for the ith body.

Dziobek's Equations

$$m_1 m_2(r_{12}^{-3} - \lambda') = \sigma A_1 A_2 \qquad m_3 m_4(r_{34}^{-3} - \lambda') = \sigma A_3 A_4$$

$$m_1 m_3(r_{13}^{-3} - \lambda') = \sigma A_1 A_3 \qquad m_2 m_4(r_{24}^{-3} - \lambda') = \sigma A_2 A_4$$

$$m_1 m_4(r_{14}^{-3} - \lambda') = \sigma A_1 A_4 \qquad m_2 m_3(r_{23}^{-3} - \lambda') = \sigma A_2 A_3$$

where λ' and μ are re-scaled Lagrange multipliers.

This leads to the famous equations of Dziobek (1900):

$$(r_{12}^{-3} - \lambda')(r_{34}^{-3} - \lambda') = (r_{13}^{-3} - \lambda')(r_{24}^{-3} - \lambda') = (r_{14}^{-3} - \lambda')(r_{23}^{-3} - \lambda')$$

Necessary and Sufficient: If these last equations are satisfied for a planar configuration, then the ratios of the masses can be obtained by dividing appropriate pairs in the first list. However, positivity of the masses must still be checked.

Relationships between the mutual distances

Requiring positivity of the masses enforces the following requirements on the mutual distances of a convex central configuration:

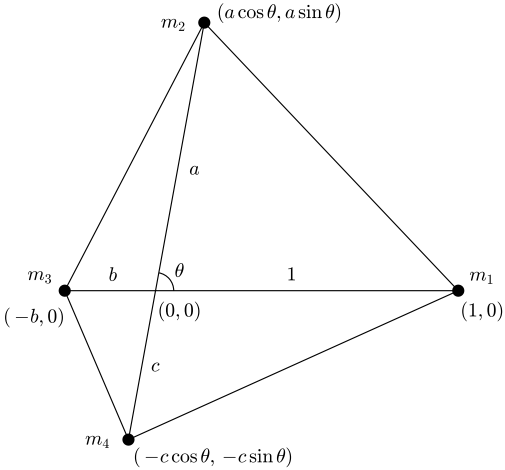
- The diagonals must be longer than all exterior sides.
- The longest and shortest exterior sides are opposite each other.

Simple Consequence: The only possible rectangular c.c. is a square, and the only possible parallelogram is a rhombus.

Assuming that the bodies are ordered sequentially in a counter-clockwise fashion and that r_{12} is the longest exterior side-length, we have the following inequalities:

$$r_{13}, r_{24} \; > \; r_{12} \; \geq \; r_{14}, r_{23} \; \geq \; r_{34}$$

Set Up: Finding Good Coordinates



Variables: $a, b, c > 0$ and $\theta \in (0, \pi)$

Symmetric Example: Kites

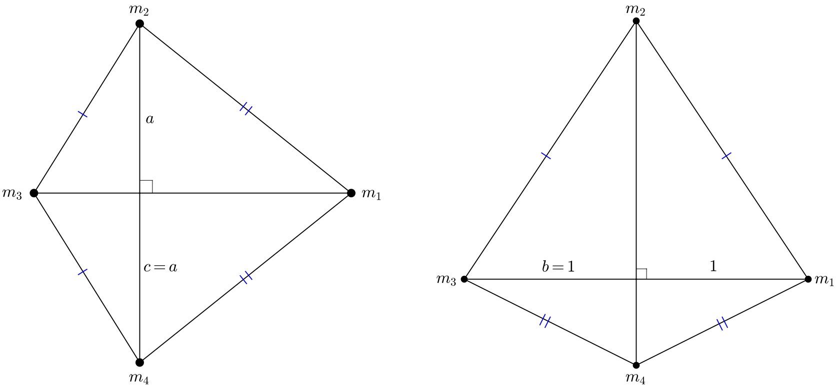


Figure: Two kite central configurations with different symmetry axes. Kites with a horizontal axis of symmetry lie in the plane $a = c$, $(m_2 = m_4)$ while those with a vertical axis of symmetry lie in the plane $b = 1$ $(m_1 = m_3)$. All kites have $\theta = \pi/2$.

Symmetric Example: Rhombus

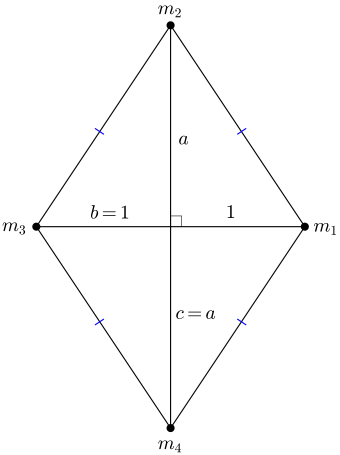


Figure: A rhombus central configuration. The rhombii occur at the intersection of the two kite planes $a = c$ and $b = 1$ (a line). We must have two pairs of opposite equal masses.

Consequences of inequalities between mutual distances

Let E denote the set of central configurations satisfying

$$r_{13}, r_{24} \;>\; r_{12} \;\geq\; r_{14}, r_{23} \;\geq\; r_{34}.$$

$$\begin{aligned} r_{12}^2 - r_{14}^2 &= (a+c)(a-c-2\cos\theta) \\ r_{23}^2 - r_{34}^2 &= (a+c)(a-c+2b\cos\theta) \end{aligned}$$

Then,

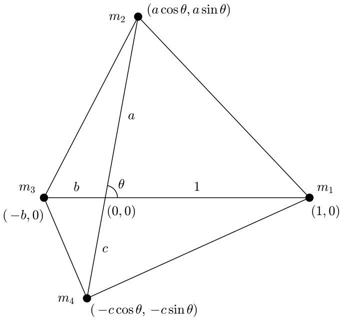
$$\begin{aligned} r_{12} \;\geq\; r_{14} \quad &\Longrightarrow \quad a-c \;\geq\; 2\cos\theta \\ r_{23} \;\geq\; r_{34} \quad &\Longrightarrow \quad a-c \;\geq\; -2b\cos\theta. \end{aligned}$$

Thus, $a \geq c$. Similarly, $r_{12} \geq r_{23}$ and $r_{14} \geq r_{34}$ imply that $b \leq 1$.

Therefore, $a = c$ and $b = 1$ (kites) are boundaries of E.

Bisecting Diagonals

What if one diagonal bisects the other?



$r_{12} \geq r_{14}$ and $r_{23} \geq r_{34}$ give

$$\frac{1}{2b}(c-a) \leq \cos\theta \leq \frac{1}{2}(a-c)$$

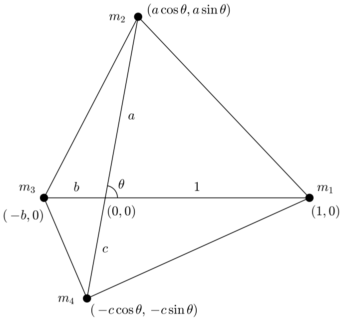
$$\frac{1}{2c}(b-1) \leq \cos\theta \leq \frac{1}{2a}(1-b)$$

Thus, $a = c$ or $b = 1$ imply $\theta = \pi/2$ (diagonals are perpendicular).

Theorem (MC, JC, GR)

A convex central configuration with one diagonal bisecting the other must be a kite.

Trapezoids



Sides $x_2 - x_1$ and $x_3 - x_4$ are parallel iff

$$\frac{a\sin\theta}{a\cos\theta - 1} = \frac{c\sin\theta}{c\cos\theta - b}$$

$$\text{iff } (ab - c)\sin\theta = 0.$$

Since $\sin\theta \neq 0$, $c = ab$.

Similarly, sides $x_2 - x_3$ and $x_1 - x_4$ are parallel iff $a = bc$. However, $a \geq c$ and $1 \geq b$ always, so $a = bc$ implies $a = c$ and $b = 1$ (rhombus).

Theorem (MC, JC, GR)

A four-body central configuration is a trapezoid if and only if $c = ab$.

Trapezoids (cont.)

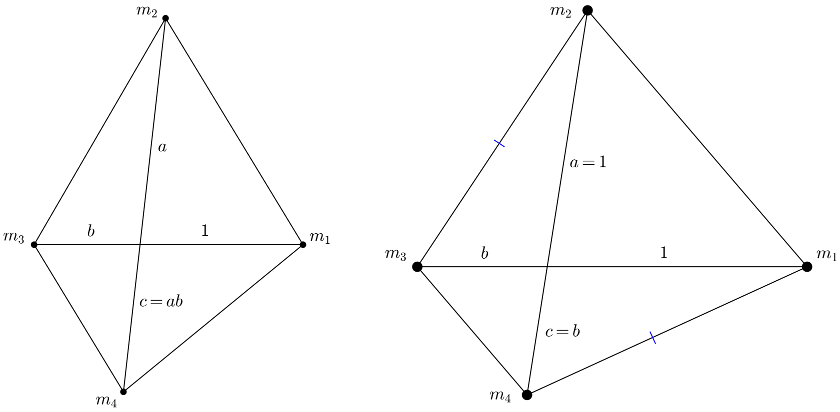
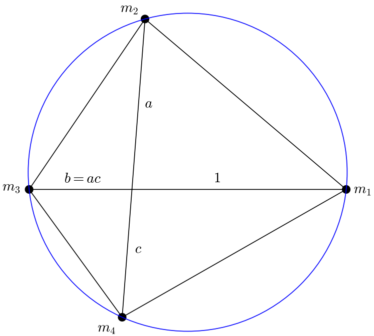


Figure: Trapezoidal configurations must satisfy $c = ab$. Assuming the configuration is not a rhombus, the parallel sides are $x_2 - x_1$ and $x_3 - x_4$. Isosceles trapezoids exist when $a = 1$ and $c = b$ (a line in *abc*-space).

Co-circular Case

The set of 4-body co-circular central configurations with positive masses is a two-dimensional surface, a graph over two of the exterior side-lengths (Cors and GR, 2012).



Theorem (MC, JC, GR)

A four-body central configuration is co-circular if and only if $b = ac$.

Proof of Co-circular Case

Recall the cross ratio from complex analysis:

$$\frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}. \tag{2}$$

The cross ratio is real if and only if the four points z_1, z_2, z_3, and z_4 lie on a circle or a line.

Setting $z_1 = 1, z_2 = ae^{i\theta}, z_3 = -b$, and $z_4 = -ce^{i\theta}$, we compute the cross ratio to be

$$\frac{(a+c)(b+1)}{ace^{i\theta} + be^{-i\theta} + a + bc}.$$

This is real if and only if $\sin\theta(ac - b) = 0$. Thus $b = ac$ is a necessary and sufficient condition for the four bodies to lie on a common circle.

Thanks to Richard Montgomery for suggesting the cross ratio.

Equidiagonal Case

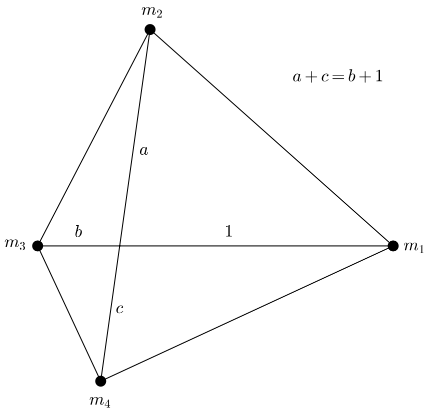


Figure: The diagonals have equal lengths if and only if $a + c = b + 1$. Examples include the isosceles trapezoids and two one-parameter families of kites.

Recap of Special Cases

- Kites: $a = c$ or $b = 1$ (two orthogonal planes)
- Rhombii: $a = c$ and $b = 1$ (a line)
- Trapezoids: $c = ab$ (a saddle)
- Isosceles Trapezoids: $a = 1$ and $b = c$ (a line)
- Co-circular configurations: $b = ac$ (a saddle)
- Equidiagonal: $a + c = b + 1$ (a plane)

Punchline: All of our classes of central configurations are defined by linear or quadratic equations. Ideal coordinates!

The Set of Convex C.C.'s

Recall Dziobek's equations:

$$(r_{12}^{-3} - \lambda')(r_{34}^{-3} - \lambda') = (r_{13}^{-3} - \lambda')(r_{24}^{-3} - \lambda') = (r_{14}^{-3} - \lambda')(r_{23}^{-3} - \lambda').$$

Eliminating λ' from the above gives

$$F \;=\; (r_{24}^3 - r_{14}^3)(r_{13}^3 - r_{12}^3)(r_{23}^3 - r_{34}^3) - (r_{12}^3 - r_{14}^3)(r_{24}^3 - r_{34}^3)(r_{13}^3 - r_{23}^3) \;=\; 0.$$

Up to an isometry, relabeling, and rescaling, the set of all four-body convex central configurations with positive masses is given by

$$E = \{(a, b, c, \theta) \in \mathbb{R}^4 : a, b, c > 0, \theta \in (0, \pi), F(a, b, c, \theta) = 0,$$

$$\text{and } r_{13}, r_{24} > r_{12} \geq r_{14}, r_{23} \geq r_{34}\}.$$

Eliminating θ

Theorem (MC, JC, GR)

Let D represent the projection of E into abc-space. For each $(a, b, c) \in D$, there exists a unique angle θ which makes the configuration central. More precisely, the set of four-body convex central configurations with positive masses is the graph of a differentiable function $\theta = f(a, b, c)$ over the domain D.

$D \subset {\mathbb{R}^+}^3$ is given by

$$0 < c \leq a,\ 0 < b \leq 1,\ \tfrac{1}{2}(-a + \sqrt{4 - 3a^2}\,) < c < \tfrac{1}{a}(b^2 + 2b),$$
$$b > \tfrac{1}{2}(-1 + \sqrt{4a^2 - 3}\,),\ \text{ and }\ c > -a + \sqrt{a^2 + b}\,.$$

These inequalities can be derived from those governing the mutual distances:

$$r_{13}, r_{24} > r_{12} \geq r_{14}, r_{23} \geq r_{34}\,.$$

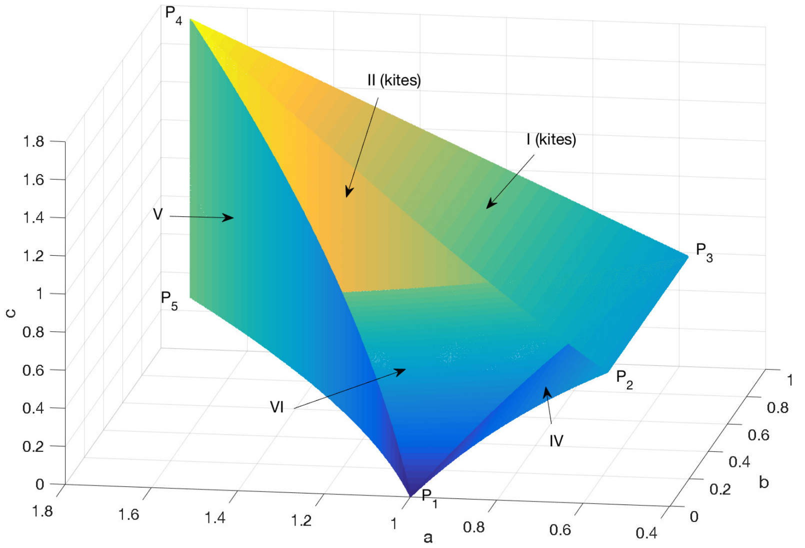


Figure: The faces of $\overline{D}$. For each point $(a, b, c) \in D$, there exists a unique angle θ that makes the corresponding configuration central.

Properties of $\overline{D}$

- $\overline{D}$ has 5 vertices, 6 faces, and 9 edges (agrees with $V - E + F = 2$).
- The kites $a = c$ and $b = 1$ are two of the faces of $\overline{D}$.
- The other faces are defined by equations such as $r_{13} = r_{12} = r_{23}$. These are degenerate cases, where one or three (not two!) of the masses vanishes (c.c.'s for the restricted problem).
- $\theta \in (\pi/3, \pi/2]$. Moreover, $\theta = \pi/2$ if and only if the configuration is a kite.
- $\overline{D}$ can be written as the union of four elementary regions in *abc*-space, where c is bounded by functions of a and b.

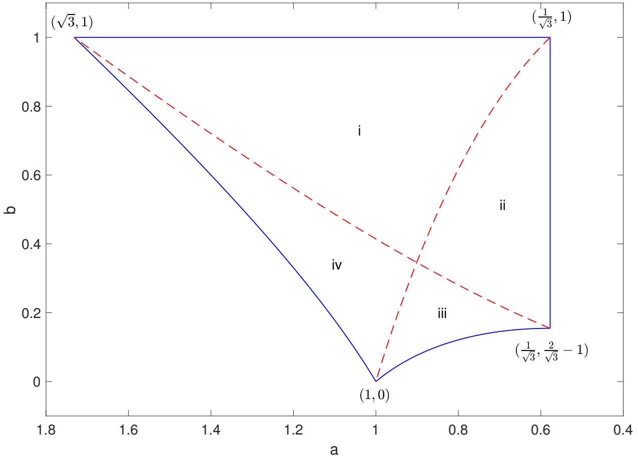


Figure: The projection of $\overline{D}$ into the *ab*-plane. The dashed red curves divide the region into four sub-regions over which *c* is bounded by functions of *a* and *b*.

Equation	Mutual Distances	Masses
$a = c$	$r_{12} = r_{14}$ and $r_{23} = r_{34}$	$m_2 = m_4$
$b = 1$	$r_{12} = r_{23}$ and $r_{14} = r_{34}$	$m_1 = m_3$
$c = \frac{1}{a}(b^2 + 2b)$	$r_{13} = r_{12} = r_{14}$	$m_2 = m_3 = m_4 = 0$
$c = \frac{1}{2}(-a + \sqrt{4 - 3a^2}\,)$	$r_{24} = r_{12} = r_{14}$	$m_3 = 0$
$b = \frac{1}{2}(-1 + \sqrt{4a^2 - 3}\,)$	$r_{13} = r_{12} = r_{23}$	$m_4 = 0$
$c = -a + \sqrt{a^2 + b}$	$r_{24} = r_{12} = r_{23}$	$m_1 = m_3 = m_4 = 0$

Table: The six faces on the boundary of D and the corresponding masses.

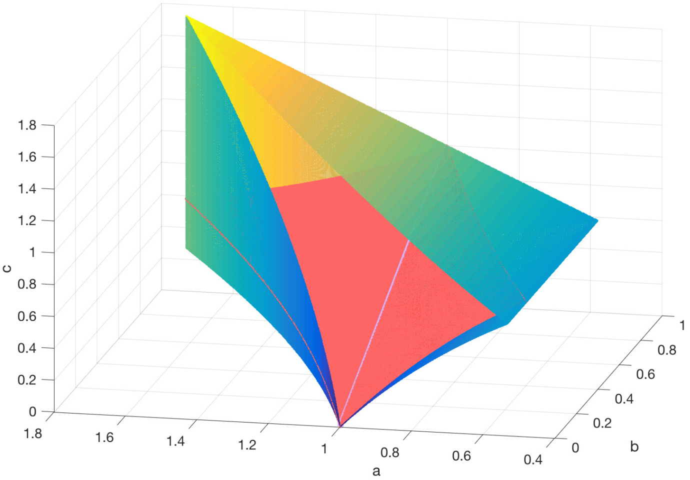


Figure: The set $\overline{D}$ along with the co-circular central configurations ($b = ac$, red).

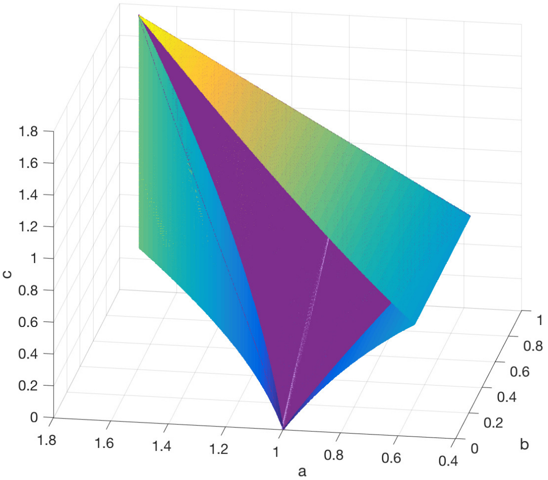


Figure: The set $\overline{D}$ along with the trapezoidal central configurations ($c = ab$, purple).

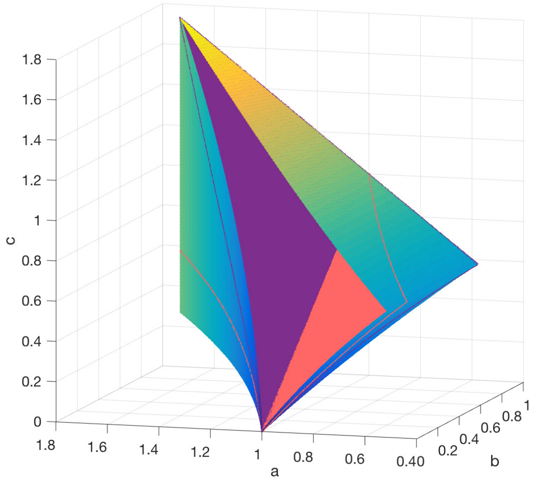


Figure: The set $\overline{D}$ along with both co-circular (red) and trapezoidal central configurations (purple).

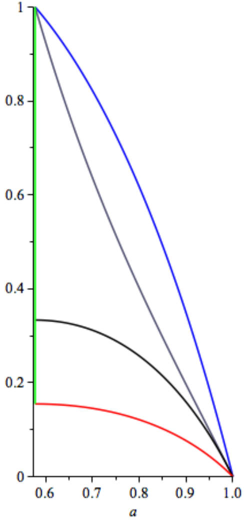


Figure: A cross-section of D with the boundary $r_{24} = r_{12} = r_{14}$ ($m_3 = 0$) showing the different classes: kites (green), equidiagonal (red), co-circular (black), trapezoids (grey), and $r_{14} = r_{23}$ (blue).

Comments/Future Work:

1. Proof of main theorem done in two parts: existence of θ accomplished by showing F has opposite signs at lower and upper bounds for θ while uniqueness follows from showing $\partial F/\partial\theta < 0$. Result then follows from the implicit function theorem.

2. We intend to use a 3d-printer to produce a model of D, perhaps including key cross sections (trapezoids, co-circular, etc.). This will give us a more intuitive understanding of the structure of D and the set of convex central configurations.

3. The hope is to use D to prove uniqueness in the masses. Little has been proven in the field about this, e.g., it is unknown whether, given a choice of masses for which a co-circular c.c. exists, that particular c.c. is unique.

4. Thank you for your attention!