On the Uniqueness of the Regular n-gon Central Configuration

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Equations of Motion

$$\begin{aligned}
\mathbf{q}_i \in \mathbb{R}^2 &= \text{position of the } i\text{-th body} \\
m_i &= \text{mass of the } i\text{-th body} \\
r_{ij} &= ||\mathbf{q}_i - \mathbf{q}_j|| \\
M &= \sum_{i=1}^{n} m_i, \quad c = \frac{1}{M}\sum_{i=1}^{n} m_i q_i \quad \text{(center of mass)} \\
U(\mathbf{q}) &= \sum_{i<j}^{n} \frac{m_i m_j}{r_{ij}} \quad \text{(Newtonian potential function)}
\end{aligned}$$

Equations of motion for the n-body problem:

$$\begin{aligned}
m_i \ddot{\mathbf{q}}_i &= \frac{\partial U}{\partial \mathbf{q}_i}, \qquad i \in \{1, 2, \ldots n\} \\
&= \sum_{j \neq i}^{n} \frac{m_i m_j (\mathbf{q}_j - \mathbf{q}_i)}{r_{ij}^3}
\end{aligned}$$

Definition

A planar *central configuration* (c.c.) is a configuration of bodies $(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n), \mathbf{x}_i \in \mathbb{R}^2$ such that the acceleration vector for each body is a common scalar multiple of its position vector (with respect to the center of mass). Specifically, in the Newtonian n-body problem with center of mass $\mathbf{c}$, for each index i, $\frac{\partial U}{\partial \mathbf{q}_i}(\mathbf{x}) = -\lambda m_i(\mathbf{x}_i - \mathbf{c})$ or

$$\sum_{j \neq i}^{n} \frac{m_i m_j (\mathbf{x}_j - \mathbf{x}_i)}{||\mathbf{x}_j - \mathbf{x}_i||^3} + \lambda\, m_i(\mathbf{x}_i - \mathbf{c}) = 0$$

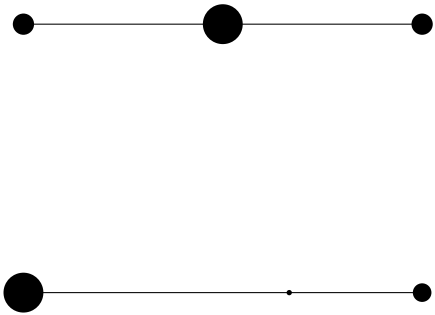
for some scalar λ independent of i.

- Finding c.c.'s is an algebra problem — no dynamics or derivatives.
- Summing together the n equations above quickly yields $\mathbf{c} = \frac{1}{M} \sum m_i \mathbf{x}_i$.

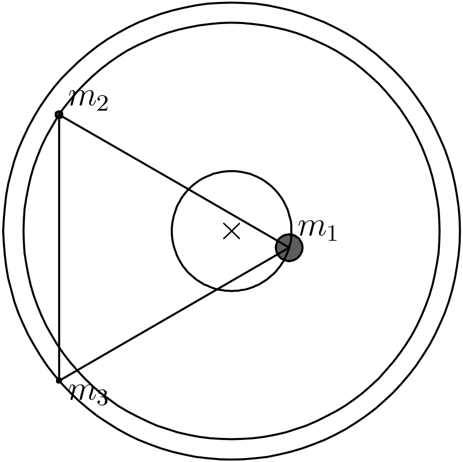
Properties of Central Configurations

- Released from rest, a c.c. maintains the same shape as it heads toward total collision (homothetic motion).
- Given the correct initial velocities, a c.c. will rigidly rotate about its center of mass. Such a solution is called a relative equilibrium.
- Any Kepler orbit (elliptic, hyperbolic, ejection-collision) can be attached to a c.c. to obtain a solution to the full n-body problem.
- For any collision orbit in the n-body problem, the colliding bodies asymptotically approach a c.c.
- Bifurcations in the topology of the integral manifolds in the planar problem (holding hc^2 constant where h is the value of the energy and c is the length of the angular momentum vector) occur precisely at values corresponding to central configurations.
- 307 articles found on MathSciNet using a general search for "central configurations"

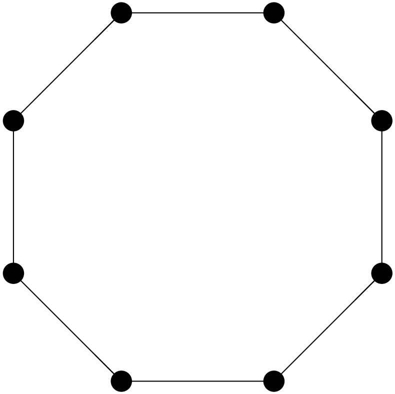
3-Body Collinear Configuration (Euler 1767)



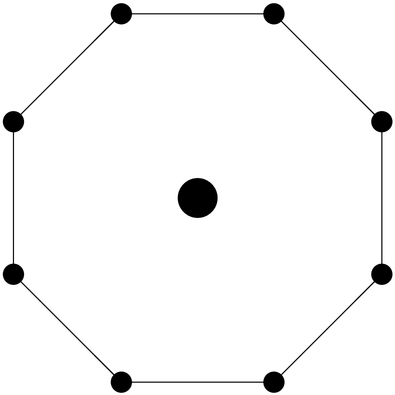
Equilateral Triangle (Lagrange 1772)



Regular n-gon (equal mass required for $n \geq 4$)



$1 + n$-gon (arbitrary central mass)



Used by Sir James Clerk Maxwell in 1859 in Stability of the Motion of Saturn's Rings (winner of the Adams Prize)

Co-Circular Central Configurations

Definition

A central configuration where the bodies all lie on a common circle is called a co-circular central configuration.

- Symmetric examples: the regular n-gon, any isosceles trapezoid, some convex kite configurations
- Asymmetric examples exist too. In the four-body problem, the set of co-circular central configurations with positive masses is a two-dimensional surface, a graph over two of the exterior side-lengths (Cors and GR, 2012).

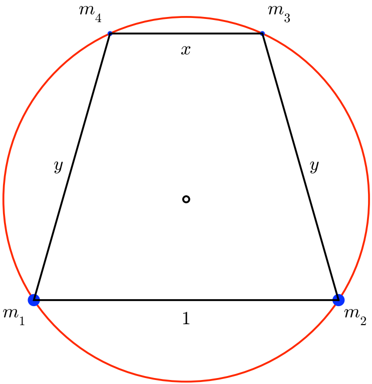


Figure : An isosceles trapezoid co-circular central configuration, where $m_1 = m_2$ and $m_3 = m_4$. The center of the circumscribing circle is marked with an O.

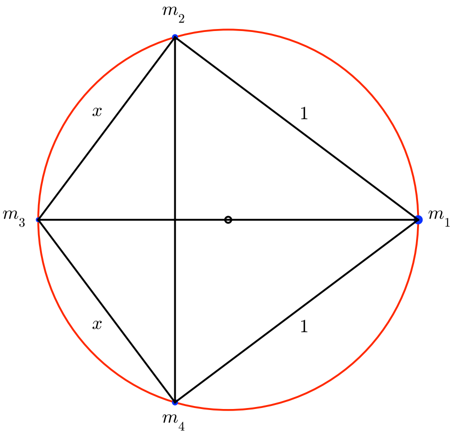


Figure : A co-circular kite central configuration, where $m_2 = m_4$. The center of the circumscribing circle is marked with an O.

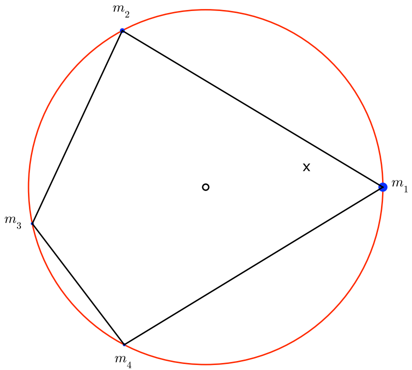


Figure : An asymmetric co-circular central configuration. The center of the circumscribing circle is marked with an O while the center of mass is labeled with an X.

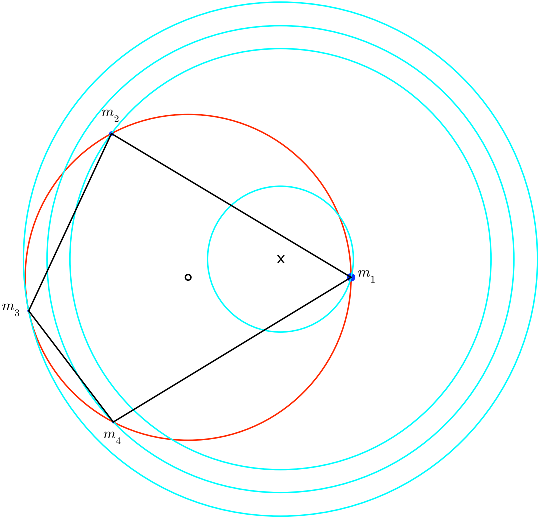


Figure : The relative equilibrium generated by the previous central configuration.

Co-Circular Central Configurations with a Special Property

Note: The regular n-gon (equal masses) has its center of mass coinciding with the center of the circle. Consequently, the corresponding relative equilibrium motion is just rotation of the n-gon along its circumscribing circle. This type of motion is called a choreography, since all the bodies are following each other around the same curve.

Question: Is the regular n-gon the *only* co-circular central configuration to have its center of mass coincide with the center of the circumscribing circle? (Alain Chenciner, 2004)

If another solution existed other than the regular n-gon, it would be a surprising example of a non-equally spaced choreography, one where the time taken for one mass to reach the position of the mass ahead of it was not constant along the configuration.

Some Results on Chenciner's Question

- $n = 4$: The only four-body co-circular central configuration with center of mass coinciding with the center of the circle is the square with equal masses (Hampton, 2003).

- $n = 5$: Llibre and Valls (2013) have announced that the regular pentagon (again with equal masses) is the only co-circular central configuration with this special property.

- Chenciner's question is listed as Problem 12 in a collection of important open problems in celestial mechanics compiled by Albouy, Cabral and Santos (2012).

Generalizing Chenciner's Question

Consider a family of potential functions U_α of the form

$$U_\alpha \; = \; \sum_{i<j} \frac{m_i m_j}{r_{ij}^{\alpha}},$$

where $\alpha > 0$ is a real parameter.

A central configuration is a special set of distinct positions $\mathbf{x}_i \in \mathbb{R}^2$ satisfying

$$\sum_{j \neq i}^{n} \frac{m_i m_j (\mathbf{x}_j - \mathbf{x}_i)}{r_{ij}^{\alpha+2}} + \frac{\lambda}{\alpha} m_i \mathbf{x}_i \; = \; 0 \qquad \text{for each } i \in \{1, \ldots, n\}$$

and for some scalar λ independent of i. Without loss of generality, we take the center of mass to be $\mathbf{c} = 0$.

The system of equations defining a central configuration can be written more compactly as

$$\nabla U_\alpha(\mathbf{x}) + \lambda \nabla I(\mathbf{x}) \;=\; 0, \tag{1}$$

where I is one half the moment of inertia, $I = \frac{1}{2}\sum_{i=1}^{n} m_i ||\mathbf{q}_i||^2$.

Note that I is a homogeneous function of degree 2 while U_α is homogeneous of degree $-\alpha$. Taking the dot product of equation (1) with $\mathbf{x}$ then yields the useful formula

$$\lambda \;\equiv\; \lambda(\alpha) \;=\; \frac{\alpha U_\alpha}{2I}.$$

Since $\alpha > 0$ and $m_i > 0$, we must have $\lambda > 0$.

Assuming the equations of motion are in the standard form $m_i \ddot{\mathbf{q}}_i = \partial U_\alpha / \partial \mathbf{q}_i$, the angular velocity of the corresponding relative equilibrium is given by $\sqrt{\lambda}$.

Angular and Radial Components

Suppose we have a central configuration on the unit circle, whose center of mass is at the origin. Let $\mathbf{x}_i = (\cos\theta_i, \sin\theta_i)$ where $\theta_i \in [0, 2\pi)$ for each i. The angles must satisfy, for each $i \in \{1, \ldots, n\}$,

$$\sum_{j \neq i}^{n} \frac{m_j}{r_{ij}^{\alpha+2}} \begin{bmatrix} \cos\theta_j - \cos\theta_i \\ \sin\theta_j - \sin\theta_i \end{bmatrix} + \frac{\lambda}{\alpha} \begin{bmatrix} \cos\theta_i \\ \sin\theta_i \end{bmatrix} = 0. \tag{2}$$

Divide these equations into angular and radial components.

The angular components are obtained by computing the dot product of the i-th equation in system (2) with the vector $[-\sin\theta_i, \cos\theta_i]^T$. This yields the system

$$\sum_{j \neq i}^{n} \frac{m_j}{r_{ij}^{\alpha+2}} \sin(\theta_j - \theta_i) = 0, \qquad \text{for each } i \in \{1, \ldots, n\}. \tag{3}$$

If equation (3) holds for some i, then the force (due to gravity for the case $\alpha = 1$) acting on the i-th body points toward the center of mass.

Angular Components (cont.)

Using the fact that

$$r_{ij} \;=\; \sqrt{2 - 2\cos(\theta_j - \theta_i)} \;=\; 2\sin\left(\frac{|\theta_j - \theta_i|}{2}\right),$$

equation (3) simplifies to

$$-\frac{1}{2^{\alpha+1}} \sum_{j \neq i}^{n} \frac{\delta_{ij}\, m_j \cos\left(\frac{\theta_j - \theta_i}{2}\right)}{\left[\sin\left(\frac{|\theta_j - \theta_i|}{2}\right)\right]^{\alpha+1}} \;=\; 0, \quad \text{where} \;\; \delta_{ij} = \begin{cases} 1 & \text{if } \theta_i - \theta_j > 0 \\ -1 & \text{if } \theta_i - \theta_j < 0. \end{cases} \tag{4}$$

Note: Equation (4) can also be derived via a variational approach. Using the principle of least action and beginning with a relative equilibrium solution on a circle with center of mass equivalent to the center of the circle, a perturbing path that only varies the angles leads to equation (4).

Radial Components

To derive the equations for the radial components, we take the dot product of the i-th equation in system (2) with the vector $[\cos\theta_i, \sin\theta_i]^T$. This gives

$$\sum_{j \neq i}^{n} \frac{m_j}{r_{ij}^{\alpha+2}} \big(\cos(\theta_j - \theta_i) - 1\big) + \frac{\lambda}{\alpha} \;=\; 0, \quad \text{for each } i \in \{1, \ldots, n\},$$

which simplifies to

$$\sum_{j \neq i}^{n} \frac{m_j}{r_{ij}^{\alpha}} \;=\; \frac{2\lambda}{\alpha}, \qquad \text{for each } i \in \{1, \ldots, n\}. \tag{5}$$

If equation (5) holds for some i, then the magnitude of the force vector acting on the i-th body is the correct length to be a central configuration.

The Key Equations

It is straight-forward to check that

$$-\frac{1}{2^{\alpha+1}} \sum_{j \neq i}^{n} \frac{\delta_{ij}\, m_j \cos\left(\frac{\theta_j - \theta_i}{2}\right)}{\left[\sin\left(\frac{|\theta_j - \theta_i|}{2}\right)\right]^{\alpha+1}} \;=\; 0, \quad \text{where} \quad \delta_{ij} = \begin{cases} 1 & \text{if } \theta_i - \theta_j > 0 \\ -1 & \text{if } \theta_i - \theta_j < 0 \end{cases}$$

and

$$\sum_{j \neq i}^{n} \frac{m_j}{r_{ij}^{\alpha}} \;=\; \frac{2\lambda}{\alpha}, \qquad \text{for each } i \in \{1, \ldots, n\}$$

define a system of $2n$ equations that are both necessary and sufficient for a central configuration on the unit circle to have its center of mass at the origin.

Suppose $\mathbf{x}_i$ now represents the position of the i-th vortex and $m_i = \Gamma_i$ is its circulation or vorticity, which may either be positive or negative. The equations of motion are determined by

$$U_0 \;=\; -\sum_{i<j} \Gamma_i \Gamma_j \ln(r_{ij}),$$

which is actually the Hamiltonian for the system.

Central configurations in the planar n-vortex problem are often referred to as stationary solutions or vortex crystals. They are found by solving

$$\sum_{j \neq i}^{n} \frac{\Gamma_i \Gamma_j (\mathbf{x}_j - \mathbf{x}_i)}{r_{ij}^2} + \omega \Gamma_i \mathbf{x}_i \;=\; 0 \quad \text{for each } i \in \{1, \ldots, n\},$$

and for some scalar ω independent of i.

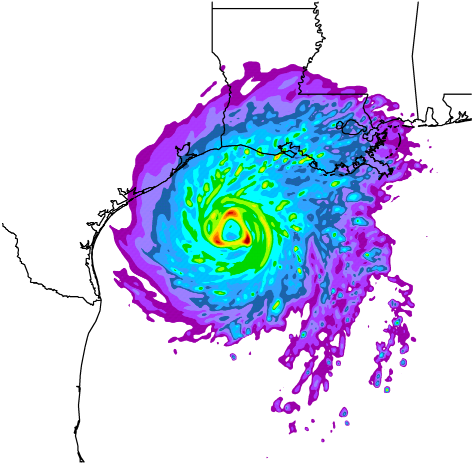


Figure : Weather research and forecasting model from the National Center for Atmospheric Research (NCAR) showing the field of precipitable water for Hurricane Rita (2005). Note the presence of three maxima near the vertices of an equilateral triangle contained within the hurricane's "polygonal" eyewall.
`http://www.atmos.albany.edu/facstaff/kristen/wrf/wrf.html`

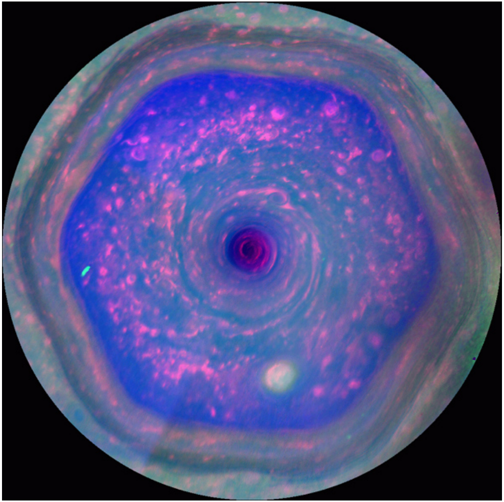


Figure : Saturn's North Pole and its encircling hexagonal cloud structure. First photographed by Voyager in the 1980's and here again recently by the Cassini spacecraft – a remarkably stable structure!

An Alternate Characterization (Vortices)

As with the generalized Newtonian case, the system of equations defining a central configuration can be written compactly as

$$\nabla U_0(\mathbf{x}) + \omega \nabla I(\mathbf{x}) \; = \; 0, \tag{6}$$

where $I = \frac{1}{2} \sum_{i=1}^{n} \Gamma_i ||\mathbf{q}_i||^2$ is now one half the angular impulse.

Taking the dot product of equation (6) with $\mathbf{x}$ yields

$$\omega \; = \; \frac{L}{2I}, \quad \text{where } L = \sum_{i<j} \Gamma_i \Gamma_j$$

is called the total angular vortex momentum.

Note that since $\Gamma_i < 0$ is now allowed, it is possible to have $\omega < 0$, which means the corresponding relative equilibrium rotates in the opposite direction. This occurs in a family of rhombi c.c.'s.

Angular and Radial Components (Vortices)

A co-circular central configuration of vortices lying on the unit circle and whose center of vorticity is at the origin satisfies

$$\sum_{j\neq i}^{n} \frac{\Gamma_j}{r_{ij}^2} \begin{bmatrix} \cos\theta_j - \cos\theta_i \\ \sin\theta_j - \sin\theta_i \end{bmatrix} + \omega \begin{bmatrix} \cos\theta_i \\ \sin\theta_i \end{bmatrix} \;=\; 0, \;\; \text{for each } i \in \{1,\ldots,n\}. \quad (7)$$

The angular components come from taking the dot product of the i-th equation in system (7) with the vector $[-\sin\theta_i, \cos\theta_i]^T$, which yields

$$\sum_{j\neq i}^{n} \frac{\Gamma_j}{r_{ij}^2} \sin(\theta_j - \theta_i) \;=\; 0, \;\; \text{for each } i \in \{1,\ldots,n\}.$$

Similarly, the radial components come from taking the dot product with $[\cos\theta_i, \sin\theta_i]^T$, which yields the surprisingly simple equation

$$\sum_{j\neq i}^{n} \Gamma_j \;=\; 2\omega, \;\; \text{for each } i \in \{1,\ldots,n\}.$$

Vortices: Equal-Strength Circulations Required

The necessary condition

$$\sum_{j \neq i}^{n} \Gamma_j \;=\; 2\omega, \quad \text{for each } i \in \{1, \ldots, n\},$$

simplifies to $\Gamma_i = \Gamma - 2\omega$ for each i, where $\Gamma = \sum_{i=1}^{n} \Gamma_i$ is the total circulation.

But $\Gamma - 2\omega$ is independent of i, so all the vorticities must be equal!

Theorem

In the planar n-vortex problem with arbitrary vorticities, a co-circular central configuration whose center of vorticity is located at the center of the circle containing the vortices must have equal-strength circulations.

Restricting U_α to the Unit Circle

Let

$$V_\alpha \;=\; V_\alpha(\theta_1, \ldots, \theta_n) \;=\; \sum_{i<j}^{n} \frac{m_i m_j}{\left[2 \sin\left(\frac{|\theta_j - \theta_i|}{2}\right)\right]^\alpha}$$

and

$$V_0 \;=\; -\sum_{i<j}^{n} \Gamma_i \Gamma_j \ln\left[2 \sin\left(\frac{|\theta_j - \theta_i|}{2}\right)\right]$$

be the restrictions of U_α and U_0, respectively, to the unit circle.

Lemma

Fix $\alpha \geq 0$ and suppose that $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ is a central configuration on the unit circle with center of mass (or vorticity) at the origin. Then the set of angles $\theta = (\theta_1, \ldots, \theta_n)$ defining the positions of $\mathbf{x}$ must be a critical point of V_α.

Motivation for the Lemma

The equations for the angular components are

$$-\frac{1}{2^{\alpha+1}}\sum_{j\neq i}^{n}\frac{\delta_{ij}\, m_j \cos\left(\frac{\theta_j-\theta_i}{2}\right)}{\left[\sin\left(\frac{|\theta_j-\theta_i|}{2}\right)\right]^{\alpha+1}} \;=\; 0, \quad \text{where} \quad \delta_{ij} = \begin{cases} 1 & \text{if } \theta_i-\theta_j>0 \\ -1 & \text{if } \theta_i-\theta_j<0. \end{cases}$$

The left-hand side of this equation is $\dfrac{1}{\alpha m_i}\cdot\dfrac{\partial V_\alpha}{\partial\theta_i}$ where

$$V_\alpha \;=\; \sum_{i<j}^{n}\frac{m_i m_j}{\left[2\sin\left(\frac{|\theta_j-\theta_i|}{2}\right)\right]^{\alpha}}.$$

Motivation for this calculation comes from a well-known but unpublished preprint by Hall (1988) on central configurations with one large mass and n small, infinitesimal masses. As the small masses approach zero, their positions limit on a circle at a critical point of a potential function such as V_α.

A Topological Approach

Theorem

Fix $\alpha \geq 0$. Given a set of positive masses m_i (or positive circulations Γ_i if $\alpha = 0$), for each ordering of the bodies on the unit circle, V_α has a unique critical point up to translation. This critical point is a minimum.

Proof Outline: Follow an approach used by Moulton for collinear c.c.'s. Suppose the bodies are arranged so that

$$0 \leq \theta_1 < \theta_2 < \cdots < \theta_n < 2\pi. \tag{8}$$

On the sub-region of $[0, 2\pi]^n$ determined by the inequalities in (8), V_α is continuous, bounded below and approaches ∞ on the boundary. Thus, V_α attains a minimum on this region.

Proof Outline cont.

To see that the critical point is unique, we examine the quadratic form $u^T D^2 V_\alpha(\theta)\, w$, where $D^2 V_\alpha$ is the Hessian matrix of V_α. This is messy, but the key feature is that

$$\frac{\partial^2 V_\alpha}{\partial \theta_i^2} \;=\; -\sum_{j \neq i}^{n} \frac{\partial^2 V_\alpha}{\partial \theta_i \partial \theta_j}.$$

For the vortex case, we compute that

$$u^T D^2 V_0(\phi)\, w \;=\; \frac{1}{4} \sum_{i<j}^{n} \Gamma_i \Gamma_j \csc^2\left(\frac{\phi_j - \phi_i}{2}\right)(u_i - u_j)(w_i - w_j).$$

Then, assuming $\Gamma_i > 0 \;\forall i$, we see that $u^T D^2 V_0(\phi)\, u \geq 0$, with equality if and only if u is a scalar multiple of $[1\,1\,\ldots\,1]^T$. A similar calculation works for the case $\alpha > 0$.

Main Result: Generalized Newtonian Case

Corollary

For any $\alpha > 0$ and for the case of equal masses, the regular n-gon is the only co-circular central configuration with center of mass coinciding with the center of the circle containing the bodies.

Easy proof: Any co-circular central configuration having its center of mass at the center of the circumscribing circle must be a critical point of V_α. By our previous theorem, since the masses are fixed, this critical point must be unique. Since the regular *n*-gon is a solution in the equal-mass case, it must be the unique solution to the problem.

Main Result: Vortex Case is Completely Solved

Corollary

In the planar n-vortex problem with arbitrary vorticities, the only co-circular central configuration with center of vorticity coinciding with the center of the circle is the regular n-gon with equal vorticities.

Easy proof: By our earlier result, the circulations of the central configuration must all be equal. Without loss of generality, we can take this common circulation to be positive. By our previous theorem, there can only be one such possible central configuration, and this has to be the regular n-gon.

Comments/Future Work:

1. Using a different approach that features a clever application of the fundamental theorem of algebra, Aref (2011) has shown that when the circulations are assumed to be equal, the only central configuration with center of vorticity coinciding with the center of the circle is the regular *n*-gon.

2. Can we use our equations to prove some special cases? Surprisingly, they don't lead to an easy proof for the case $n = 4$ (e.g., with Gröbner bases). Maybe they help with symmetric configurations?

3. What about connecting the vortex case ($\alpha = 0$) to the Newtonian case ($\alpha = 1$), treating α as a smooth parameter? Can we prove that there is no bifurcation as α increases? Then the result for the vortex case would have to extend to all $\alpha > 0$.

4. Thank you for your attention!