Elusive Zeros Under Newton's Method

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Newton's Method

Iterative root-finding method $f(x) = 0$: $x_0, x_1, x_2, \ldots$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

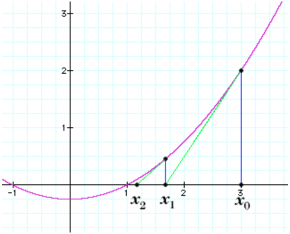

Figure: Newton's Method for finding a root of a function on $\mathbb{R}$. Image source: http://aleph0.clarku.edu/~djoyce/newton/method.html

Newton's Method as a Dynamical System

$$N_p(z) = z - \frac{p(z)}{p'(z)}, \qquad p : \mathbb{C} \mapsto \mathbb{C}$$

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- Newton's method "tends" to obey the *nearest-root principal*: initial seeds iterate towards the closest root.
- If $p(z)$ is a quadratic polynomial with distinct roots, N_p is topologically conjugate to $z \mapsto z^2$. The Julia set of N_p is precisely the perpendicular bisector of the line segment connecting the two roots.

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- Ugly: In certain cases, Newton's method N_p may contain an extraneous attracting cycle distinct from the roots of p. This would yield an entire open **region** of the complex plane that never converges to a root. Here, a small perturbation may not improve your situation!

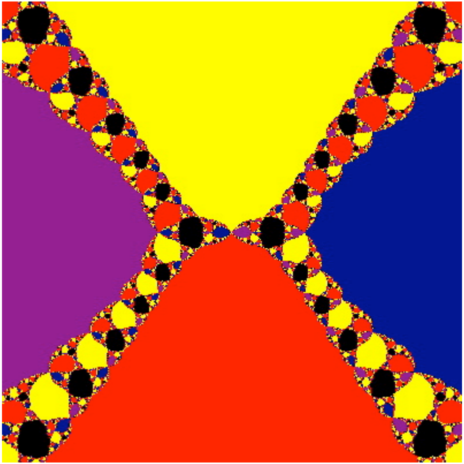

Figure: The dynamical plane for Newton's method applied to $p_\lambda(z) = (z-1)(z+1)(z-\lambda)(z-\bar{\lambda})$ with $\lambda \approx 0.4438656912\,i$. The "bad" initial seeds (black) iterate towards a super-attracting period 2-cycle.

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Since $N_p'(z) = \dfrac{p(z)\,p''(z)}{[p'(z)]^2}$, the inflection points of p are the free critical points of N_p.

The Cubic Case

$$p_\lambda(z) \;=\; (z-1)(z+1)(z-\lambda), \quad \lambda \in \mathbb{C}$$

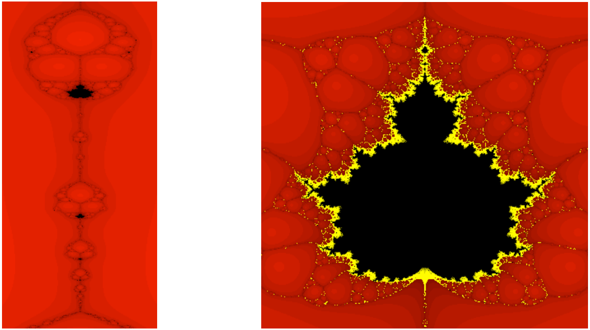

Figure: The parameter plane for Newton's method applied to p_λ. Black parameter values correspond to polynomials for which the free critical point does not converge to a root, ie., it is drawn into an extraneous attracting cycle.

Research on Cubic Newton Maps

- J. Curry, L. Garnett and D. Sullivan (1983)
- J. Head (1988)
- S. Sutherland (1989)
- Tan Lei (1990, 1997)
- F. Haesler and H. Kriete (1993)
- P. Blanchard (1994)
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- Theory of polynomial-like mappings
  A. Douady and J. Hubbard (1985)

A Symmetric Fourth-Degree Polynomial Family

$$\begin{aligned} p_\lambda(z) &= (z-1)(z+1)(z-\lambda)(z-\bar{\lambda}), \quad \lambda \in \mathbb{C} \\ &= z^4 - 2\mathrm{Re}(\lambda)z^3 + (|\lambda|^2 - 1)z^2 + 2\mathrm{Re}(\lambda)z - |\lambda|^2 \end{aligned}$$

Symmetric location of the roots (kite configuration) leads to nice reductions and interesting dynamics.

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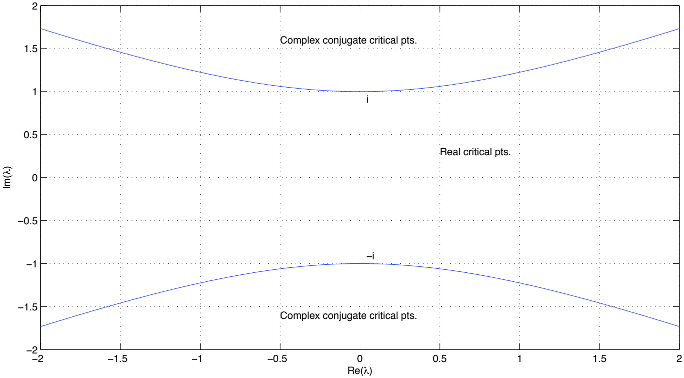
Two free critical points: $p_\lambda'' = 0$

$$
c_\pm = \frac{1}{2}\left(\mathrm{Re}(\lambda) \pm \sqrt{(\mathrm{Re}(\lambda))^2 - \frac{2}{3}(|\lambda|^2 - 1)}\right)
$$

Goal: Follow the orbits of $c_\pm$ as λ varies. If an extraneous attracting cycle exists, it must attract at least one of these orbits.

The Parameter Plane



If $\lambda = a + bi$, then the discriminant of the quadratic defining the two critical points $c_{\pm}$ is given by

$$\delta \;=\; \frac{1}{3}\left(a^2 - 2b^2 + 2\right).$$

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- Real axis is invariant under N_λ
- For $\lambda \in \mathbb{R}$, $c_\pm$ converge to a root of p_λ (analytic proof)
- For $\lambda = \beta i$, $N_{\beta i} \sim N_{i/\beta}$. For this interesting case, we can restrict to a complicated 1-d real map with $0 < \beta \leq 1$ (analytic work)

The Case $\lambda = \beta i$

$$N_\beta(x) = \frac{3x^4 + (\beta^2 - 1)x^2 + \beta^2}{4x^3 + 2(\beta^2 - 1)x}.$$

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- For $\beta_2 = (2\sqrt{5} - 3)/\sqrt{11} \approx 0.4438656912$, c_+ and c_- lie on a super-attracting 2-cycle.
- For odd periods, the free critical points can never lie on the *same* periodic orbit.

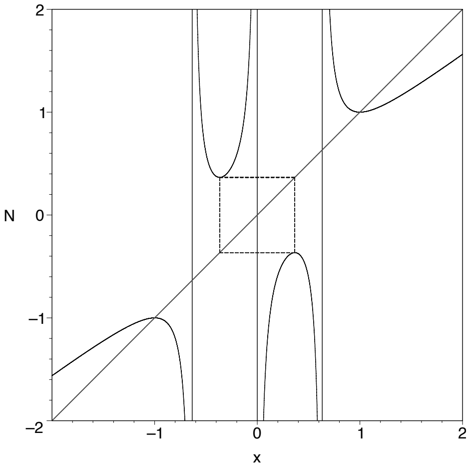
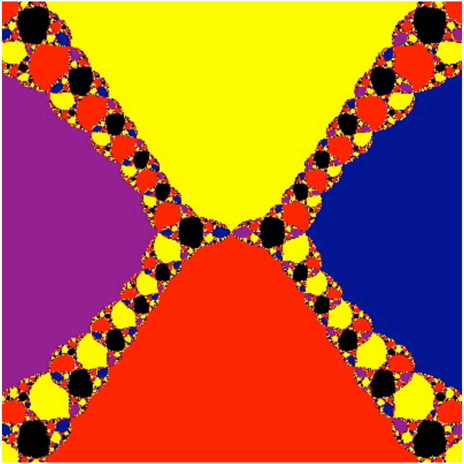


Figure: The orbit diagram for N_β with $\beta = (2\sqrt{5} - 3)/\sqrt{11} \approx 0.4438656912$ showing a super-attracting 2-cycle between c_+ and c_-.



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Per	β	Type	Per	β	Type
2	0.4438657165	Bitransitive	5	0.2296915054	Disjoint
2	0.3835689425	Disjoint	5	0.2275660932	Disjoint
3	0.2291103601	Disjoint	5	0.2249682546	Disjoint
3	0.1341462433	Disjoint	5	0.1846443415	Disjoint
4	0.3642913699	Disjoint	5	0.1577119529	Disjoint
4	0.3363839984	Disjoint	5	0.1301919222	Disjoint
4	0.2158225775	Bitransitive	5	0.1289675832	Disjoint
4	0.2113012969	Disjoint	5	0.1125293225	Disjoint
4	0.1134351641	Disjoint	5	0.0917167962	Disjoint
4	0.0616595671	Disjoint	5	0.0570865125	Disjoint
5	0.2299712598	Disjoint	5	0.0298646167	Disjoint

Table: The table of β values for which N_β has super-attracting periodic cycles. Also listed is the type of cycle: Bitransitive (free critical points on same orbit) or Disjoint (free critical points on separate orbits)

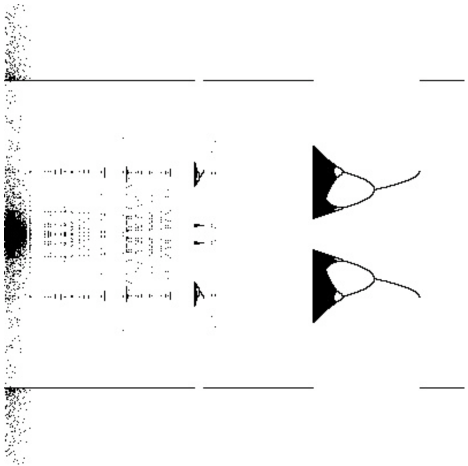

Figure: The bifurcation diagram for N_β showing the asymptotic behavior of both free critical points as a function of β. The horizontal line segments at the top and bottom of the figure are 1 and -1.

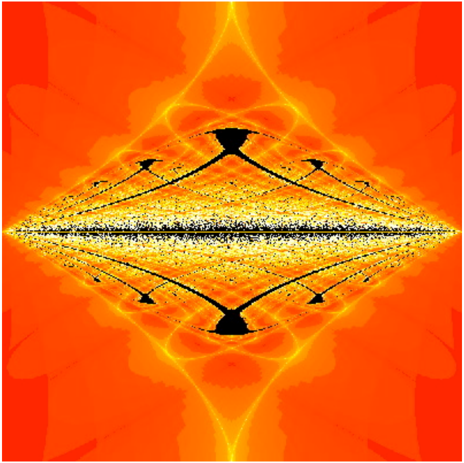

Figure: The λ-parameter plane for N_λ following the orbit of both free critical points (shading indicates different rates of convergence.) The window is $[-1, 1] \times [-i, i]$.

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- **Bitransitive:** Critical points attracted to same periodic orbit. Obtain *swallow configurations* and *tricorns* in a real cross-section of the parameter plane. Prototype models:
  Swallow: $x \mapsto (x^2 + c_1)^2 + c_2, \quad c_1, c_2 \in \mathbb{R}$
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- **Disjoint Periodic Sinks:** Critical points attracted to different periodic orbits. Obtain *product configurations* and *Mandelbrot sets* in a real cross-section of the parameter plane. Prototype models:
  Product: $x \mapsto x^2 + c_1, y \mapsto y^2 + c_2, \quad c_1, c_2 \in \mathbb{R}$
  Mandelbrot Set: $z \mapsto z^2 + c, \quad c \in \mathbb{C}$

Figure: An example of Milnor's "swallow configuration" in the parameter plane for N_λ centered at the bitransitive value $\lambda \approx 0.443865i$.

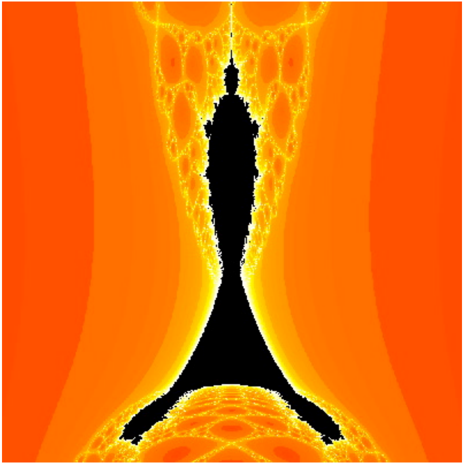

Figure: As expected (according to Milnor), a tricorn is located in the parameter plane at the inversion $(1/\beta)\, i$ of the bitransitive value of the previous figure. In this case, the two free critical points are complex conjugates. The prototype for this case is the map $z \mapsto (z^2 + c)^2 + \bar{c}$.

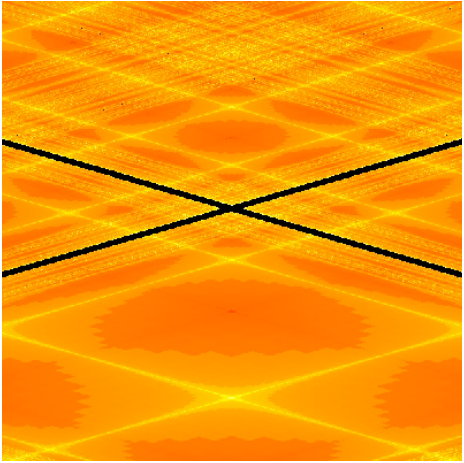

Figure: Zooming in on the parameter plane near the a disjoint periodic value, $\lambda \approx 0.2291i$, exhibiting a “product” configuration.

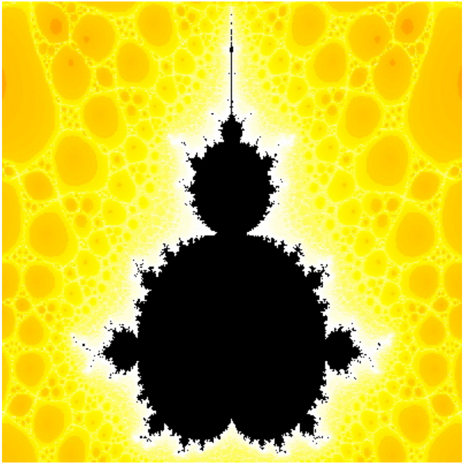

Figure: The Mandelbrot-like set in the parameter plane arising from the inversion $(1/\beta)\, i$ of our disjoint periodic value of the previous figure.

Some Final Observations

- Conjecture: Each bitransitive λ-value corresponding to the two free critical points sharing the same super-attracting n-cycle lies at the center of a swallow configuration in the parameter plane.

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- The yellow diamond shaped boundary in the parameter plane is defined by those λ-values where both p'_λ and p''_λ simultaneously vanish. If $\lambda = a + bi$, this occurs on the algebraic curve

$$(a^2 - 2b^2 + 2)^3 - 27a^2(b^2 + 1)^2 = 0.$$

  Taking successive pre-images of this curve appears to define the sequence of intertwining yellow "leaves" that approach the real axis.

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Thank You for Your Attention