Morse Theory and Stability of Relative Equilibria in the Planar n-Vortex Problem

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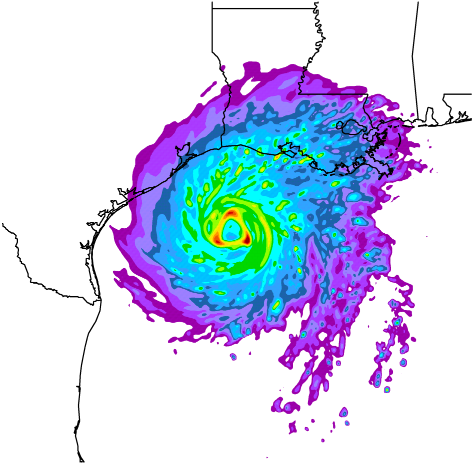


Figure: Weather research and forecasting model from the National Center for Atmospheric Research (NCAR) showing the field of precipitable water for Hurricane Rita (2005). Note the presence of three maxima near the vertices of an equilateral triangle contained within the hurricane's "polygonal" eyewall.

`http://www.atmos.albany.edu/facstaff/kristen/wrf/wrf.html`

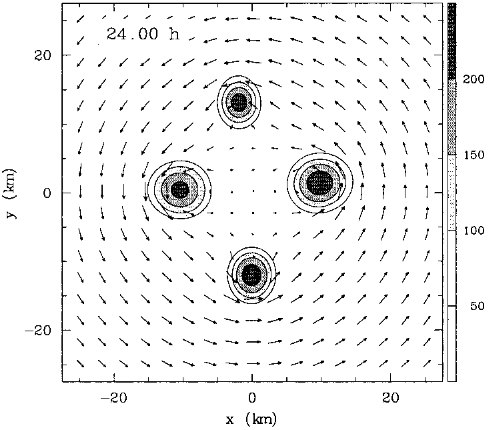


Figure: Result of a numerical simulation carried about by Kossin and Shubart to model the evolution of very thin annular rings of high vorticity ("Mesovortices, Polygonal Flow Patterns, and Rapid Pressure Falls in Hurricane-Like Vortices," Kossin and Shubert, *Journal of Atmospheric Sciences*, 2001.) Note the "vortex crystal" of four vortices located close to a rhombus configuration. Darker shading indicates higher vorticity. The flow pattern shown lasted for about 18 hours.

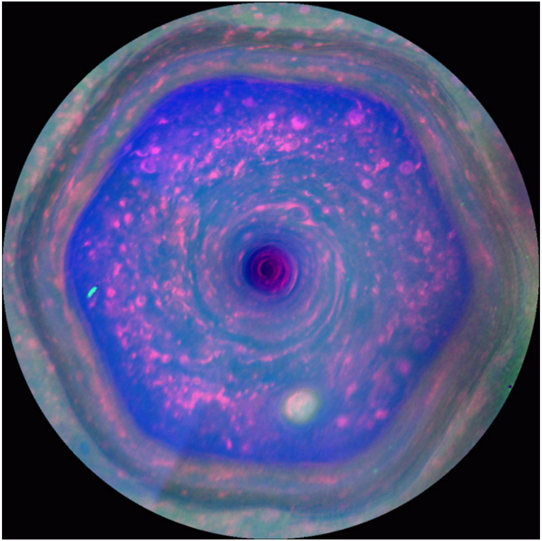


Figure: Saturn's North Pole and its encircling hexagonal cloud structure. First photographed by Voyager in the 1980's and here again recently by the Cassini spacecraft – a remarkably stable structure!

Description of the n-Vortex Problem

- Introduced by Helmholtz (1858) to model a two-dimensional slice of columnar vortex filaments. Later refined by Lord Kelvin (1867) and Kirchoff (1876).
- Widely used model providing finite-dimensional approximations to vorticity evolution in fluid dynamics.
- General goal is to track the motion of the point vortices rather than focus on their internal structure and deformation, a concept analogous to the use of "point masses" in celestial mechanics.
- Generally "easier" than the n-body problem, e.g., the planar three-vortex system is integrable.
- Many techniques used to study the n-body problem work perfectly well (sometimes even better) in the n-vortex problem.

A system of n planar point vortices with vortex strength $\Gamma_i \neq 0$ and positions $x_i \in \mathbb{R}^2$ evolves according to

$$\Gamma_i \dot{x}_i \;=\; J\,\frac{\partial H}{\partial x_i} \;=\; J \sum_{j \neq i}^{n} \frac{\Gamma_i \Gamma_j}{r_{ij}^2}(x_j - x_i), \quad 1 \leq i \leq n$$

where

$$H = -\sum_{i<j} \Gamma_i \Gamma_j \ln(r_{ij}), \quad r_{ij} = \|x_i - x_j\|, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The *configuration space* is $\mathbb{R}^{2n} - \Delta$, where

$$\Delta \;=\; \{(x_1, \ldots, x_n) : x_i = x_j \;\text{ for some } i \neq j\}$$

is the set of collisions.

Special Solutions: Relative Equilibria

Definition

A *relative equilibrium* is a periodic solution of the form

$$x_i(t) = c + e^{-\omega J t}(x_i(0) - c), \quad 1 \le i \le n,$$

that is, a uniform rotation with angular velocity $\omega \neq 0$ around some point $c \in \mathbb{R}^2$.

The initial positions $x_i(0)$ must satisfy

$$-\omega \Gamma_i (x_i(0) - c) = \frac{\partial H}{\partial x_i} = \sum_{j \neq i}^{n} \frac{\Gamma_i \Gamma_j}{r_{ij}^2}(x_j(0) - x_i(0)), \quad 1 \le i \le n.$$

If the *total circulation* $\Gamma = \sum_i \Gamma_i \neq 0$, then the center of rotation c must be the *center of vorticity*, $c = \frac{1}{\Gamma}\sum_i \Gamma_i x_i$.

A Topological View

The *angular impulse* (with respect to the center of vorticity) is the quantity

$$I \;=\; \sum_{i=1}^{n} \Gamma_i \|x_i - c\|^2,$$

the analog of the *moment of inertia* in the n-body problem. I is a conserved quantity in the planar n-vortex problem.

Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^{2n}$ be a vector of initial positions. Then the equations defining a relative equilibrium can be written more compactly as

$$\nabla H(x) + \frac{\omega}{2} \nabla I(x) \;=\; 0,$$

where ∇ is the usual gradient operator.

Key Fact: A relative equilibrium is a critical point of H restricted to a level surface of I, with $\omega/2$ serving as the Lagrange multiplier.

Symmetries

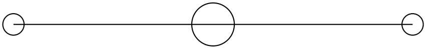
Suppose that $x \in \mathbb{R}^{2n}$ is a relative equilibrium. The following are also relative equilibria:

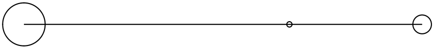
1. $kx = (kx_1, \ldots, kx_n)$ for any $k > 0$ (scaling; $c \mapsto kc, \omega \mapsto \omega/k^2$)
2. $x - s = (x_1 - s, \ldots, x_n - s)$ for any $s \in \mathbb{R}^2$ (translation; $c \mapsto c - s$)
3. $Ax = (Ax_1, \ldots, Ax_n)$ where $A \in SO(2)$ (rotation; $c \mapsto Ac$)

Thus, relative equilibria are not isolated. It is standard practice to fix a scaling and center of vorticity c, and then identify solutions that are equivalent under a rotation.

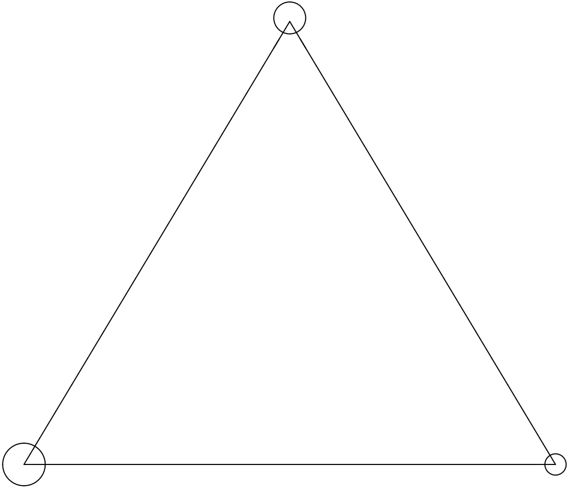
Note: Reflections of x are also relative equilibria (e.g., multiplying the first coordinate of c and each x_i by -1), but these are regarded as distinct solutions.

3-Vortex Collinear Configurations (Gröbli 1877)

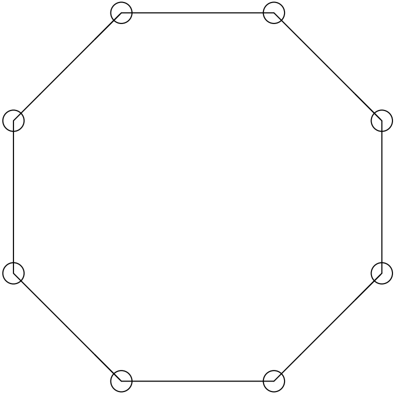




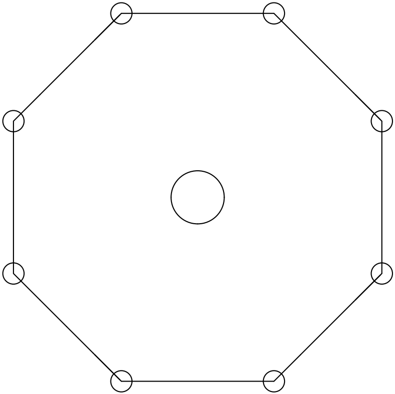
Equilateral Triangle (Lord Kelvin 1867, Gröbli 1877)



Regular n-gon (equal vorticities required for $n \geq 4$)



$1 + n$-gon (arbitrary central vortex)



Goals

Assume that $\Gamma_i > 0 \ \forall i$ and suppose that $x \in \mathbb{R}^{2n} - \Delta$ is a relative equilibrium. Then x is a critical point of the smooth function H subject to the constraint $I = I_0$ (an ellipsoid).

The *Morse index* is the dimension of the maximal subspace for which the Hessian of $H + (\omega/2)I$ at x is negative definite.

1. What is the connection between the Morse index of x and the stability of the corresponding relative equilibrium periodic solution?

2. What, if anything, do the Morse inequalities reveal? Is there a way to get some "easy" stability results using the Morse inequalities without having to laboriously compute the eigenvalues of a particular relative equilibrium?

Our study builds on the work of Moeckel, Palmore, Smale, Shub, Buck, Conley, Pacella, Hampton, Santoprete, ...

Results

Suppose that $\Gamma_i > 0 \ \forall i$ and that x is a relative equilibrium. Let $\mathrm{ind}(x)$ be the Morse index of x.

Theorem (GR)

The Morse index of x is equal to the number of real (nonzero) pairs $\pm\lambda_j$ of eigenvalues of the corresponding relative equilibrium.

Corollary (GR; 2013)

A relative equilibrium is linearly stable if and only if it is a nondegenerate minimum of H subject to the constraint $I = I_0$.

Theorem (Palmore, 1982)

$$\mathrm{ind}(x) \ \leq \ n - 2$$

The upper-bound is attained at collinear configurations.

$m = 2/5$

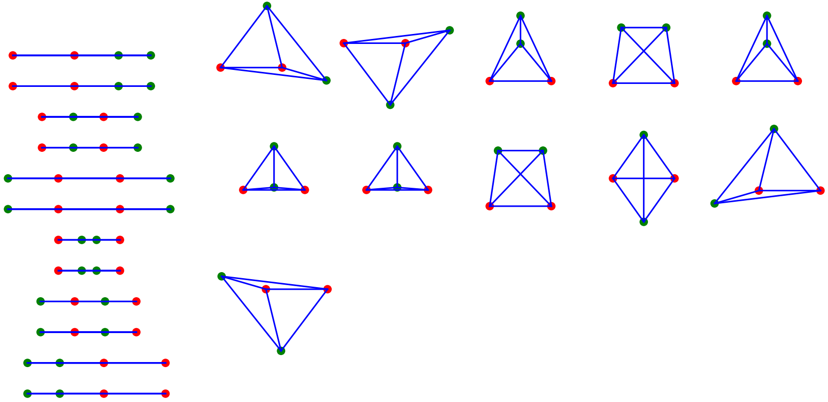


Figure: The complete set of relative equilibria (34 solutions) in the four-vortex problem with circulations $\Gamma_1 = \Gamma_2 = 1$ (red) and $\Gamma_3 = \Gamma_4 = m = 2/5$ (green). The 12 collinear solutions have index 2, the 16 concave configurations (kite, asymmetric) have index 1 and the 6 convex configurations (trapezoid, rhombus) have index 0. This result holds for all $m \in (0, 1)$.

The Details

Recall that $I = \sum_{i=1}^{n} \Gamma_i \|x_i - c\|^2$. We restrict to the *normalized configuration space*

$$\mathcal{N} \; = \; \{x \in \mathbb{R}^{2n} : c = 0, I(x) = 1\},$$

which eliminates the translational invariance and fixes the scaling. $\mathcal{N}$ is diffeomorphic to $\mathbf{S}^{2n-3}$.

If x is a critical point of $H|_{\mathcal{N}}$, then so is Ax for any $A \in \mathrm{SO}(2)$. To eliminate the rotational symmetry, we work on the quotient manifold

$$\mathcal{M} \; = \; (\mathcal{N} - \Delta)/\mathrm{SO}(2)$$

of dimension $2n - 4$. A relative equilibrium is *nondegenerate* if it is a nondegenerate critical point of H restricted to $\mathcal{M}$.

For three vortices, $\mathcal{N}/\mathrm{SO}(2)$ is diffeomorphic to $\mathbf{S}^2$ and is called the *shape sphere*, since it represents the space of all triangles up to translation, scaling, and rotation. $\mathcal{M} = (\mathcal{N} - \Delta)/\mathrm{SO}(2)$ is thus the shape sphere minus three points.

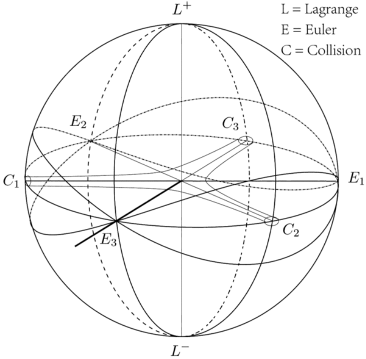


Figure: The shape sphere. H has five critical points: two equilateral triangles at the North and South Poles (minima) and three collinear configurations on the equator (saddles).

Dealing with Collisions

Problem: H blows up on Δ (collision set), but the space $\mathcal{N} - \Delta$ is not compact. Would like to work away from Δ.

If $\Gamma_1 = \Gamma_2 = \Gamma_3 = \Gamma_4 = 1$ and $\Gamma_5 = -1/2$, there exists a continua of relative equilibria which does limit on Δ.

Theorem (GR)

For a fixed choice of circulations $\Gamma_i > 0$, there exists a neighborhood of Δ in $\mathcal{N}$ which does not contain any relative equilibria.

1. The corresponding fact in the n-body problem is known as Shub's Lemma, but Shub's proof does not generalize to the vortex setting, nor does the recent argument of Moeckel.
2. For mixed-sign circulations, if $\displaystyle\sum_{\substack{i<j \\ i,j\in\Lambda}} \Gamma_i\Gamma_j \neq 0$ for all possible subsets of indices Λ, then the same result holds.

Suppose that $x \in \mathcal{N}$ is a relative equilibrium. The Hessian of $H|_{\mathcal{N}}$ at x can be obtained by restricting the $2n \times 2n$ matrix

$$G(x) \;=\; D^2 H(x) + \omega M$$

to the tangent space of $\mathcal{N}$, where $M = \mathrm{diag}\{\Gamma_1, \Gamma_1, \ldots, \Gamma_n, \Gamma_n\}$. This is easier than working in local coordinates.

Since M is positive definite, we can work with the *modified Hessian*

$$M^{-1} G(x) \;=\; M^{-1} D^2 H(x) + \omega I,$$

where I is the $2n \times 2n$ identity matrix. By *Sylvester's Inertia Law*,

$$\mathrm{ind}(x) \;=\; \text{\# of negative eigenvalues of } M^{-1} G(x)$$

Note: One can show using the homogeneity of H and I that

$$\omega \;=\; L/I \;=\; L,$$

where $L \;=\; \sum_{i<j} \Gamma_i \Gamma_j$ is the *total vortex angular momentum*.

Trivial Eigenvalues

Define $K = \mathrm{diag}\{J, J, \ldots, J\}$ where $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

1. Due to conservation of the center of vorticity, the vectors $s = [1, 0, 1, 0, \ldots, 1, 0]^T$ and Ks are in the kernel of $M^{-1}D^2H(z_0)$. They are orthogonal to the tangent space of $\mathcal{N}$. Consequently, the modified Hessian has the eigenvalue $L > 0$ repeated twice.

2. If $x \in \mathcal{N}$ is a relative equilibrium, then the vector Kx is in the kernel of both the Hessian and modified Hessian. This follows from the rotational symmetry. The vector x, which is orthogonal to the tangent space of $\mathcal{N}$, produces an eigenvalue of $2L > 0$.

Punchline: The modified Hessian always has the *trivial eigenvalues* $L, L, 2L, 0$. These are excluded when restricting to the manifold $\mathcal{M}$ (dimension $2n - 4$).

The Stability Matrix

By writing the equations of motion in rotating coordinates, the linear stability of x is determined by the eigenvalues of the *stability matrix*

$$B(x) \;=\; K(M^{-1}D^2H(x) + LI) \;=\; KM^{-1}G(x).$$

Key idea: We know how K acts on the modified Hessian because $D^2H(x)\,K \;=\; -K\,D^2H(x)$. It follows that $\mathbb{R}^{2n}$ splits into n invariant subspaces of the form $\{v_j, Kv_j\}$. This is the same splitting for either the index or stability calculations.

Punchline: The characteristic polynomial of B factors completely as

$$p(\lambda) \;=\; \lambda^2(\lambda^2 + L^2)\prod_{j=1}^{n-2}(\lambda^2 + L^2 - \mu_j^2),$$

where the μ_j are the nontrivial eigenvalues of $M^{-1}D^2H(x)$.

Theorem (GR)

The Morse index of x is equal to the number of real (nonzero) pairs $\pm\lambda_j$ of eigenvalues of the corresponding relative equilibrium.

(i) Eigenvalues of $M^{-1}G(x)$ come in pairs $(\gamma_j, 2L - \gamma_j)$, which is symmetric w.r.t. $L > 0$. We can assume $\gamma_j \leq L$ for each j.

(ii) $\pm\sqrt{\gamma_j(\gamma_j - 2L)}$ are the eigenvalues of the stability matrix $B(x)$.

Case	Eval. Pair of $M^{-1}G(x)$ (index)	Eval. pair of $B(x)$ (stability)
1.	$(-,+)$	real pair (instability)
2.	$(0,+)$	degenerate
3.	$(+,+)$	pure imaginary pair (stability)

Note: For pure imaginary eigenvalues $\pm i\beta_j$, we have $\beta_j \leq L = \omega$.

The Morse Inequalities

Recall that a relative equilibrium x is a critical point of H restricted to the manifold $\mathcal{M} = (\mathcal{N} - \Delta)/\mathrm{SO}(2)$. The *Morse inequalities* relate the indices of the critical points to the topology of $\mathcal{M}$:

$$\sum_k \alpha_k t^k = \sum_k \beta_k t^k + (1+t)Q(t),$$

where α_k is the number of critical points of index k, β_k is the k-th Betti number (the rank of the homology group $H_k(\mathcal{M}, \mathbb{R})$), and $Q(t)$ is a polynomial with non-negative integer coefficients.

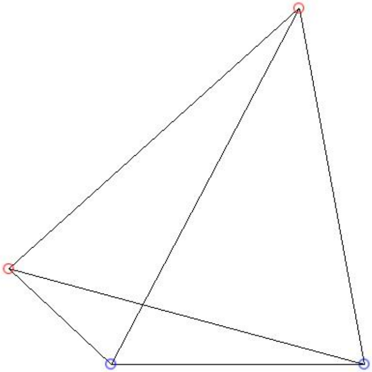
Theorem (Moeckel)

The Poincaré polynomial for $\mathcal{M} = (\mathcal{N} - \Delta)/SO(2)$ is

$$P(t) = (1+2t)(1+3t)\cdots(1+(n-1)t).$$

$n = 3$: Shape sphere minus 3 points. $P(t) = 1 + 2t$ so $\beta_0 = 1$, $\beta_1 = 2$.

Example: Two pairs of equal strength vortices



Animation: The asymmetric family of relative equilibria: $\Gamma_1 = \Gamma_2 = 1$ (blue) and $\Gamma_3 = \Gamma_4 = m$ (red), with $-1 < m \leq 1$. The configuration is concave for $m > 0$ and convex for $m < 0$.

Example: Two pairs of equal strength vortices

Set $\Gamma_1 = \Gamma_2 = 1$, $\Gamma_3 = \Gamma_4 = m$, with $0 < m < 1$. There are exactly 34 distinct relative equilibria (HRS, 2013):

- 6 convex configurations (isosceles trapezoid, rhombus)
- 16 concave configurations (kites, asymmetric)
- 12 collinear configurations

$$\gamma_0 + \gamma_1 t + \gamma_2 t^2 \;=\; 1 + 5t + 6t^2 + (1+t)(r_0 + r_1 t) \tag{1}$$

We know $\gamma_0 \geq 6$ (trapezoid and rhombus) and $\gamma_2 \geq 12$ (collinear), so we have $r_0 \geq 5$ and $r_1 \geq 6$. Setting $t = 1$ in (1) gives

$$34 = \gamma_0 + \gamma_1 + \gamma_2 = 12 + 2(r_0 + r_1) \quad \text{or} \quad r_0 + r_1 = 11.$$

Thus, $r_0 = 5$ and $r_1 = 6$, which implies $\gamma_0 = 6, \gamma_1 = 16$, and $\gamma_2 = 12$.

Punchline: The remaining relative equilibria (kites, asymmetric) have index 1 and hence, one pair of real eigenvalues (unstable).

Final Thoughts

- For H to be a Morse function, the critical points must be nondegenerate. The Morse inequalities do not hold unless nondegeneracy is confirmed. But this is a hard problem! The equilateral triangle with a central vortex ($m = 1$) is quite degenerate, with nullity equal to 3. The number of relative equilibria drops from 34 to 26 when $m = 1$.

- Future work: Apply the same techniques to relative equilibria of the four-vortex problem with three equal circulations (e.g., $\Gamma_1 = \Gamma_2 = \Gamma_3 = 1, \Gamma_4 = m$).

- Is it possible to extend this theory to the setting where vortex circulations have opposite signs? In this case the level surface $I = 1$ becomes a hyperboloid, so the topology changes dramatically. The circulation matrix M is not positive definite, so the key matrix $M^{-1} D^2 H(x)$ no longer behaves nicely (e.g., it may have complex eigenvalues).