The Planar, Circular, Restricted Four-Body Problem

Gareth E. Roberts Julianne Kulevich Christopher J. Smith

Department of Mathematics and Computer Science
College of the Holy Cross
Worcester, MA USA

NSF DMS-0708741

Joint Mathematics Meetings
San Diego, CA
January 6–9, 2008

The Equilateral Triangle Solution of Lagrange

Lagrange (1772): Place three bodies, of any masses, at the vertices of an equilateral triangle and apply the appropriate velocities to obtain a special periodic solution. Each body traces out a circle centered at the center of mass of the triangle. The shape and size of the configuration is preserved during the motion.

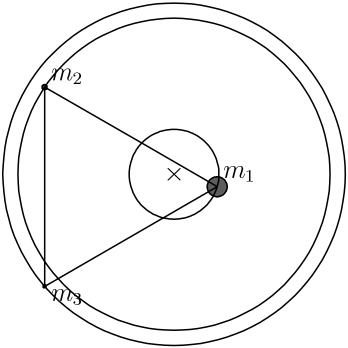

Figure: The equilateral triangle solution of the three-body problem.

The Planar, Circular, Restricted Four-Body Problem (PCR4BP)

Insert a fourth infinitesimal mass that has no influence on the circular orbits of the three larger masses ("primaries"). Change to a rotating coordinate system in a frame where the primaries are fixed. Let (x, y) be coordinates for the infinitesimal mass in this new frame.

Equations of motion: (assume $m_1 + m_2 + m_3 = 1$)

$$\begin{aligned} \ddot{x} &= 2\dot{y} + V_x \\ \ddot{y} &= -2\dot{x} + V_y \end{aligned}$$

where

$$V(x, y) = \frac{1}{2}\left((x - c_x)^2 + (y - c_y)^2\right) + \frac{m_1}{a} + \frac{m_2}{b} + \frac{m_3}{c}$$

is the **amended potential**, (c_x, c_y) is the center of mass of the primaries and a, b, c represent the respective distances of the infinitesimal mass from each of the three primaries.

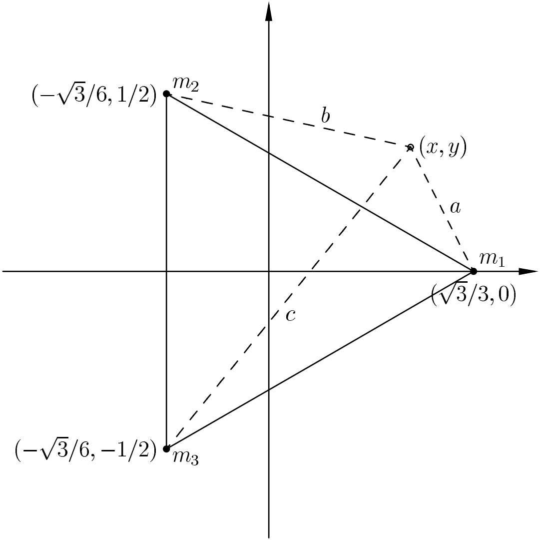

Figure: Setup for the planar, circular, restricted four-body problem.

Two Finiteness Questions

$$V(x,y) = \frac{1}{2}\left((x-c_x)^2 + (y-c_y)^2\right) + \frac{m_1}{a} + \frac{m_2}{b} + \frac{m_3}{c}$$

$$\begin{aligned} \ddot{x} &= 2\dot{y} + V_x \\ \ddot{y} &= -2\dot{x} + V_y \end{aligned}$$

Let $\dot{x} = u, \dot{y} = v$. Integral of motion:

$$E = \frac{1}{2}(u^2 + v^2) - V \quad \text{(Jacobi)}$$

Note: Critical points of V are equilibrium points of the PCR4BP ("parking" spaces).

1. How do the location and number of critical points change as the masses of the primaries are varied? Are there a **finite** number of critical points for **all** choices of m_1, m_2 and m_3?
2. Is it possible for a solution to the above equations to travel along a level curve of V? (*Saari's Conjecture*)

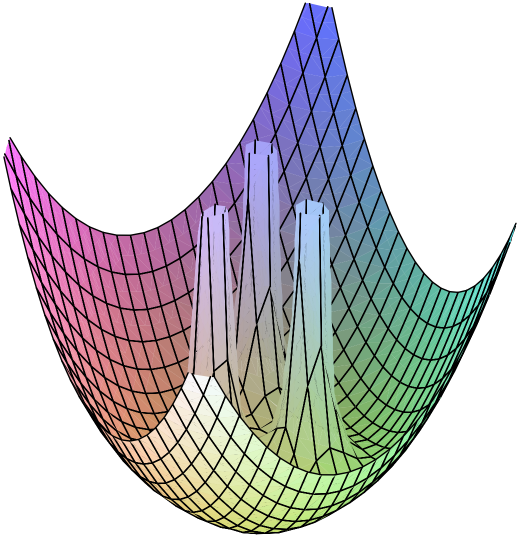

Figure: The amended potential V for the case of three equal masses.

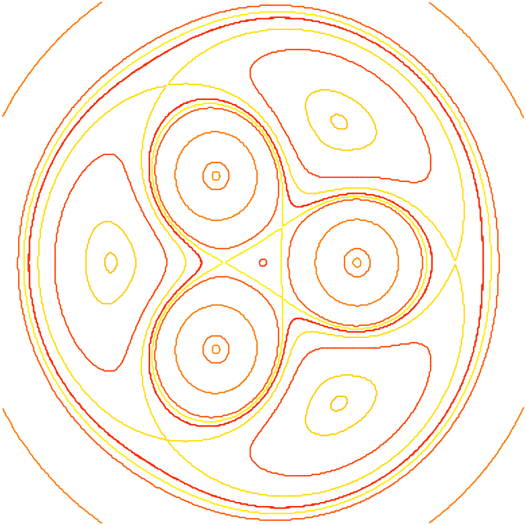

Figure: Level curves of the amended potential V for the case of three equal masses. There are 10 critical points — 6 saddles and 4 minima.

Using Distance Coordinates

Treat the distances a, b, c as variables:

$$x = \frac{\sqrt{3}}{6}\left(b^2 + c^2 - 2a^2\right) \quad y = \frac{1}{2}\left(c^2 - b^2\right)$$

subject to the constraint

$$F = a^4 + b^4 + c^4 - (a^2b^2 + a^2c^2 + b^2c^2) - (a^2 + b^2 + c^2) = -1$$

(Cayley-Menger Determinant). In these new coordinates, the amended

potential function becomes

$$V = \frac{1}{2}\left(m_1 a^2 + m_2 b^2 + m_3 c^2\right) + \frac{m_1}{a} + \frac{m_2}{b} + \frac{m_3}{c} + \text{constant.}$$

Equations for the Critical Points

$$m_1\left(1 - \frac{1}{a^3}\right) + 2\lambda(2a^2 - b^2 - c^2 - 1) = 0 \quad (1)$$

$$m_2\left(1 - \frac{1}{b^3}\right) + 2\lambda(2b^2 - a^2 - c^2 - 1) = 0 \quad (2)$$

$$m_3\left(1 - \frac{1}{c^3}\right) + 2\lambda(2c^2 - a^2 - b^2 - 1) = 0 \quad (3)$$

$$a^4 + b^4 + c^4 - (a^2b^2 + a^2c^2 + b^2c^2) - (a^2 + b^2 + c^2) = -1$$

Summing equations $(1), (2)$ and (3) yields

$$\lambda = \frac{1}{6}\left(1 - \frac{m_1}{a^3} - \frac{m_2}{b^3} - \frac{m_3}{c^3}\right).$$

Eliminating λ

$$2a^5b^3c^3 - 2m_3a^5b^3 - 2m_2a^5c^3 - a^3b^5c^3 + m_3a^3b^5 - a^3b^3c^5$$
$$+(3m_1 - 1)a^3b^3c^3 + m_3a^3b^3c^2 + m_3a^3b^3 + m_2a^3b^2c^3 + m_2a^3c^5$$
$$+m_2a^3c^3 - 2m_1a^2b^3c^3 + m_1b^5c^3 + m_1b^3c^5 - 2m_1b^3c^3 = 0$$

$$2a^3b^5c^3 - 2m_3a^3b^5 - 2m_1b^5c^3 - a^5b^3c^3 + m_3a^5b^3 - a^3b^3c^5$$
$$+(3m_2 - 1)a^3b^3c^3 + m_3a^3b^3c^2 + m_3a^3b^3 + m_1a^2b^3c^3 + m_1b^3c^5$$
$$+m_1b^3c^3 - 2m_2a^3b^2c^3 + m_2a^5c^3 + m_2a^3c^5 - 2m_2a^3c^3 = 0$$

$$a^4 + b^4 + c^4 - (a^2b^2 + a^2c^2 + b^2c^2) - (a^2 + b^2 + c^2) + 1 = 0$$

Symmetry: $a \leftrightarrow b, m_1 \leftrightarrow m_2$

Equal Mass Case

Theorem

(GR, JK, CS 2007) The number of critical points in the PCR4BP for equal masses is exactly 10.

Proof: Due to the equal masses, it is possible to show that all critical points must lie on an altitude of the equilateral triangle ($a = b$, $a = c$ or $b = c$). This reduces the problem down to two equations in two unknowns. Using Gröbner bases (or resultants), we obtain a 22 degree polynomial that contains 5 positive real roots. Of these 5, three correspond to physically relevant solutions of the original equations. By symmetry, this gives a total of 9 critical points. The 10th is found at the origin, where all three altitudes intersect.

Remark: This result is subsumed by numerical and analytic work of Pedersen (1944), Simó (1978), Arenstorf (1982) and Leandro (2006).

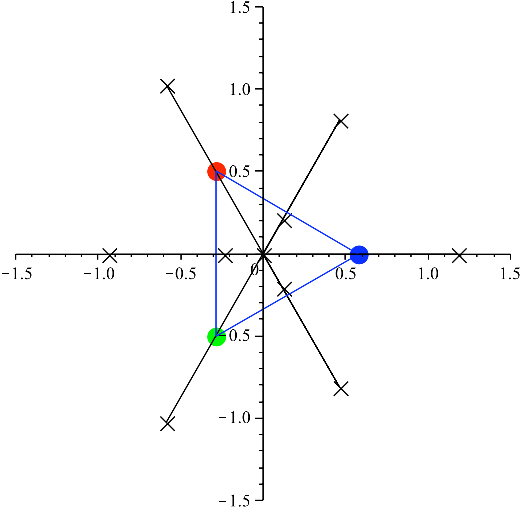

Figure: The 10 equilibria for the PCR4BP in the case of equal masses. Note the symmetry with respect to the equilateral triangle of the primaries.

Theorem

(GR, JK, CS 2007) The number of critical points in the PCR4BP is finite for any choice of masses. In particular, there are less than 268 critical points.

Remark: Our result showing finiteness appears to be new. The upper bound of 268 is not optimal as the work of Pedersen (1944), Simó (1978), Arenstorf (1982) and Leandro (2006) suggests the actual number varies between 8 and 10. It is a surprisingly complicated problem to study the bifurcation curve in the mass parameter space for which there are precisely 9 critical points.

BKK Theory

Bernstein, D. N., The Number of Roots of a System of Equations, *Functional Analysis and its Applications*, **9**, no. 3, 183-185, 1975.

Given $f \in \mathbb{C}[z_1, \ldots z_n]$, $f = \sum c_k z^k$, $\quad k = (k_1, k_2, \ldots, k_n)$, the **Newton polytope** of f, denoted $N(f)$, is the convex hull in $\mathbb{R}^n$ of the set of all exponent vectors occurring for f.

Given $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \in \mathbb{Q}$, the **reduced polynomial** f_α is the sum of all terms of f whose exponent vectors k satisfy

$$\alpha \cdot k = \min_{l \in N(f)} \alpha \cdot l.$$

This equation defines a face of the polytope $N(f)$ with inward pointing normal α.

Let $\mathbb{T} = (\mathbb{C}^*)^n$ where $\mathbb{C}^* = \mathbb{C} - \{0\}$.

Theorem

(Bernstein, 1975) Suppose that system (4) has infinitely many solutions in $\mathbb{T}$. Then there exists a vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \in \mathbb{Q}$ and $\alpha_j = 1$ for some j, such that the system of reduced equations (5) also has a solution in $\mathbb{T}$ (all components nonzero).

$$\begin{aligned} f_1(z_1, \ldots, z_n) &= 0 \\ f_2(z_1, \ldots, z_n) &= 0 \\ &\vdots \\ f_m(z_1, \ldots, z_n) &= 0, \end{aligned} \tag{4}$$

$$\begin{aligned} f_{1\alpha}(z_1, \ldots, z_n) &= 0 \\ f_{2\alpha}(z_1, \ldots, z_n) &= 0 \\ &\vdots \\ f_{m\alpha}(z_1, \ldots, z_n) &= 0. \end{aligned} \tag{5}$$

The "big Minkowski"

Key Fact: Bernstein argues that it is sufficient to check a finite number of vectors α since different vectors can induce the **same** reduced equations. Using the Minkowski sum polytope

$$N(f_1) + N(f_2) + \cdots + N(f_m) = \{v \in \mathbb{R}^n : v = v_1 + \cdots + v_m, v_i \in N(f_i)\},$$

only the inward normals of each facet of this "big Minkowski" need be considered. We must also examine the reduced equations for "faces" of codimension greater than one. If all such α's fail to yield a nontrivial solution (all components nonzero), then Bernstein's theorem shows that the number of solutions to the system is finite.

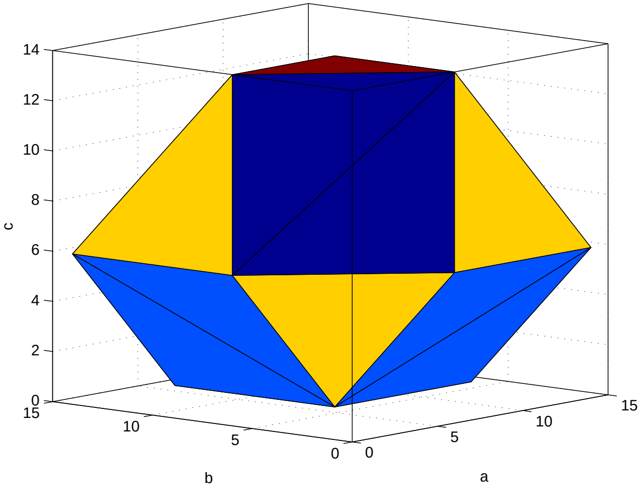

Figure: The Minkowski sum polytope corresponding to the three equations for the critical points of V.

Good Example

Choose $\alpha =< 0, 1, 1 >$

Reduced equations:

$$\begin{aligned} -2m_3a^5b^3 - 2m_2a^5c^3 + m_3a^3b^3 + m_2a^3c^3 &= 0 \\ m_3a^5b^3 + m_2a^5c^3 + m_3a^3b^3 - 2m_2a^3c^3 &= 0 \\ a^4 - a^2 + 1 &= 0 \end{aligned}$$

Gröbner basis: $\{a^4 - a^2 + 1, m_3b^3, m_2c^3\}$

No solutions in $\mathbb{T}$ means this α is excluded. Yay!

Difficult Example

Choose $\alpha =< 1, 0, 0 >$

Reduced equations:

$$\begin{aligned} m_1 b^5 c^3 + m_1 b^3 c^5 - 2m_1 b^3 c^3 &= 0 \\ -2m_1 b^5 c^3 + m_1 b^3 c^5 + m_1 b^3 c^3 &= 0 \\ b^4 + c^4 - b^2 c^2 - b^2 - c^2 + 1 &= 0 \end{aligned}$$

Problem: $b = \pm 1, c = \pm 1$ and $a \neq 0$ is a nontrivial solution to the reduced equations. Boo! Bernstein's Theorem doesn't help.

Puiseux Series

Hampton, M. and Moeckel, R., Finiteness of relative equilibria of the four-body problem, *Inventiones mathematicae* **163**, 289-312, 2006.

Puiseux series (complex) :

$$x(t) = \sum_{i=i_0}^{\infty} a_i t^{\frac{i}{q}} \;, \quad q \in \mathbb{N},\; i_0 \in \mathbb{Z}$$

If a system of n polynomial equations has an infinite variety in $\mathbb{T}$, then there exists a convergent Puiseux series solution $x_j(t), j = 1, \ldots n$ with **order** α. Moreover, one of the variables is simply $x_l(t) = t$.

The **order** of the Puiseux series solution is the vector of rationals arising from the fractional exponent of the first term in each series. This vector α is precisely the same α of Bernstein's theorem.

Good News

- Minkowski sum polytope for our system has 14 facets, 12 vertices and 24 edges.
- Using symmetry and ignoring those inward normals with all coordinates non-positive, only two inward normals remain that have reduced equations with nontrivial solutions: $\alpha_1 =< 1, 0, 0 >$ and $\alpha_2 =< 0, 0, 1 >$.
- For each "bad" α, we can substitute Puiseux series in t into the original equations ($a = t$ for α_1 and $c = t$ for α_2), and show that no such series solution can exist by examining higher order terms in t (Implicit Function Theorem).
- Of the 15 edges that need to be examined, most have reduced equations with either no solution or a trivial solution. The others can be eliminated using symmetry.

Final Remarks and Future Work

- The vertices of the Minkowski sum polytope (faces of codimension 3) are quickly eliminated since they yield at least one reduced equation with a single monomial, and thereby a trivial solution. This completes the proof of finiteness.
- The lower bound of 268 for the number of critical points is obtained by computing the **mixed volume** of the polytopes corresponding to our system of equations (Bernstein).
- What about solutions traveling on level curves? Hopefully similar techniques will show this is impossible, but polynomials are much, much larger (over 10,000 terms).
- Additional problem: PCRnBP with equal mass primaries on a regular *n*-gon. Applications to the charged *n*-body problem?