Archimedes' Quadrature of the Parabola

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Outline

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- Perhaps most telling: we do know he designed a tombstone for himself illustrating the discovery he wanted most to be remembered for (discussed by Plutarch, Cicero)

Figure: Sphere inscribed in cylinder of equal radius

$3V_{sphere} = 2V_{cyl}$ and $A_{sphere} = A_{cyl}$ (lateral area)

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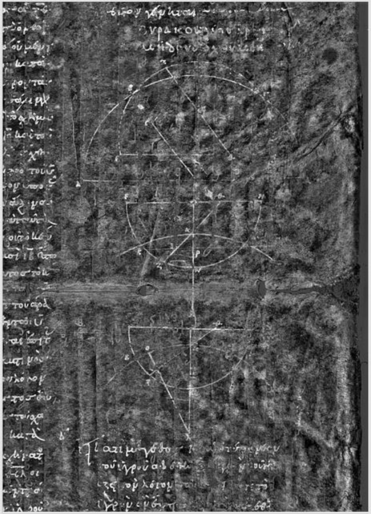
- 1906 – a *palimpsest* prayerbook created about 1229 CE (a reused manuscript) was found by to contain substantial portions (a 10th century CE copy from older sources)

A page from the palimpsest



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- Example: Letter to Eratosthenes (in Alexandria) at start of *The Method*

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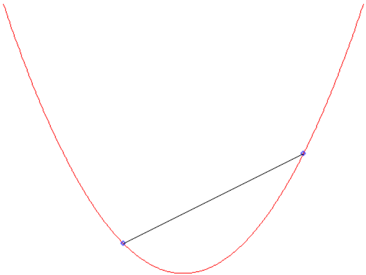
- Get a glimpse of the way Archimedes thought about what he was doing

What is the area of a parabolic segment?



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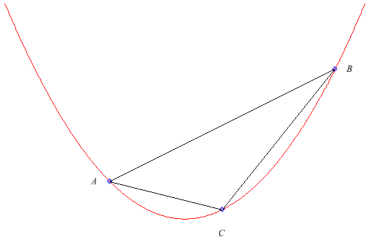

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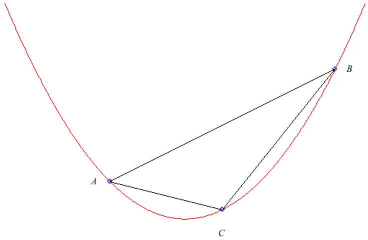

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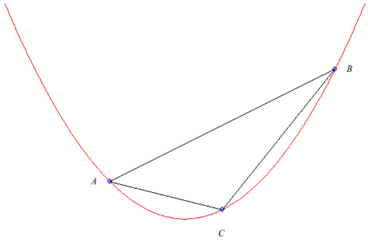

- Let C have same x-coordinate as the midpoint of AB
- The area of the parabolic segment $= \frac{4}{3}\text{area}(\Delta ABC)$

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- We will look at both of these using modern notation – both quite interesting!

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- The point C is called the *vertex* of the segment

Sketch of Archimedes' first proof

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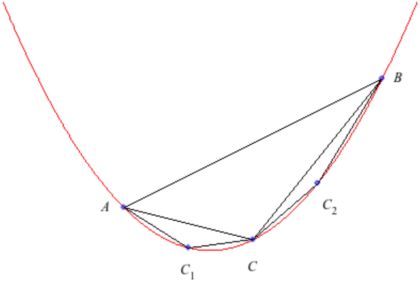

The first proof, continued

- Archimedes shows that each of those triangles has area $\frac{1}{8}$ the area of ΔABC, so

$$\text{area}(\Delta AC_1C) + \text{area}(\Delta CC_2B) = \frac{1}{4}\text{area}(\Delta ABC).$$

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- We do this construction some finite number n of times and add together the areas of all the triangles,

First proof, continued

- letting A_0 be the original area, $A_1 = \frac{1}{4}A_0$ the area of the first two smaller triangles, then $A_2 = \frac{1}{4}A_1 = \frac{1}{4^2}A_0$ the area of the next four, etc.

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- Moreover, $\frac{4}{3}A_0 - S_n = \frac{1}{3}A_n$, so as n increases without bound, S_n tends to $\frac{4}{3}A_0$

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- We would derive all this, of course, using the usual properties of finite and infinite *geometric series*:

$$\text{area of segment} = \sum_{n=0}^{\infty} \frac{A_0}{4^n} = \frac{A_0}{1 - \frac{1}{4}} = \frac{4A_0}{3}.$$

Archimedes had to do it all in an *ad hoc* way because that general theory did not exist yet!

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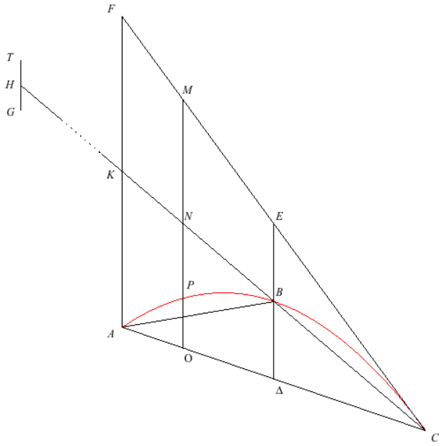
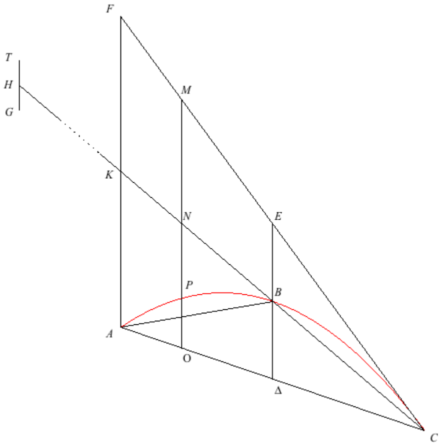
- Key Idea: Apply a physical analogy – will compute the area of the parabolic segment essentially by considering it as a thin plate, or "lamina," of constant density and "weighing it on a balance" in an extremely clever way

The construction



- *Note:* Δ in figure $=$ D in text (a Maple issue!)
- CF is tangent to the parabola at C
- B is the vertex
- O is an arbitrary point along the line from A to C
- ED, MO, FA are parallel to the axis of the parabola

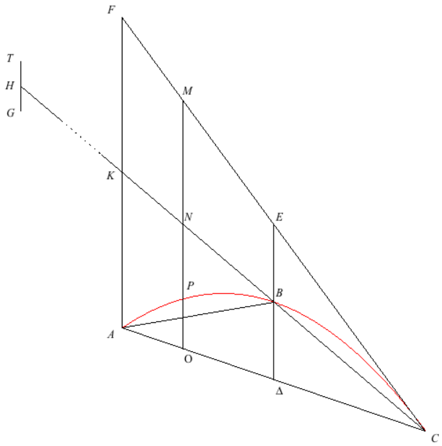
First observation



- By construction of the vertex B, $\text{area}(\Delta ADB) = \text{area}(\Delta CDB)$
- "Well-known" property of parabolas (at least in Archimedes' time!): $BE = BD$, $NM = NO$, $KA = KF$
- $\therefore$ $\text{area}(\Delta EDC) = \text{area}(\Delta ABC)$ and by similar triangles,

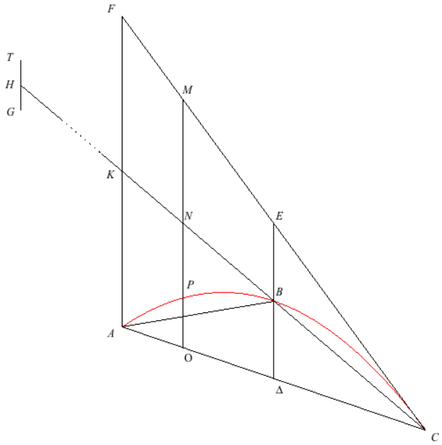
$$\text{area}(\Delta AFC) = 4 \cdot \text{area}(ABC)$$

Next steps



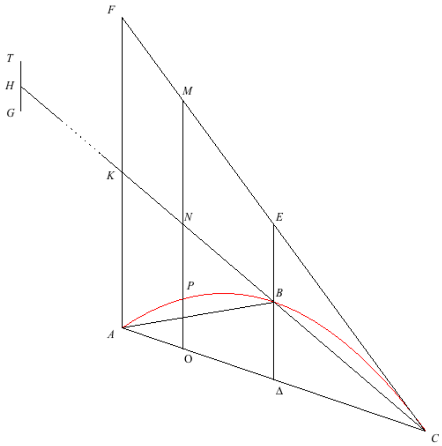
- Produce CK to H making $CK = KH$
- Consider CH "as the bar of a balance" with K at fulcrum
- Known facts about triangles (proved by Archimedes earlier): The *centroid* of ΔAFC is located at a point W along CK with $CK = 3 \cdot KW$.

"Dissection" step



- Consider PO (think of a thin strip approximating the part of the parabolic segment with some thickness Δx)
- Properties of parabolas and similar triangles imply $MO : PO = AC : AO = CK : KN = HK : KN$
- Place segment $TG = PO$ with midpoint at H.
- Then Archimedes' *law of the lever* (fulcrum at K) says TG balances MO

Conclusion of the proof



- Do this for all "strips" like PO. We get that ΔACF exactly balances the whole collection of vertical strips making up the parabolic segment.
- By the law of the lever again, area(ΔACF) : area of segment $= HK : KW = 3 : 1$
- Since area(ΔACF) $=$ $4 \cdot$ area(ΔABC), the proof is complete.

From the introductory letter of *The Method* to Eratosthenes

"Seeing moreover in you, as I say, an earnest student, a man of considerable eminence in philosophy, and an admirer of mathematical inquiry, I thought fit to write out for you and explain in detail ... a certain method by which it will be possible for you to get a start to enable you to investigate some of the problems in mathematics by means of mechanics. This procedure is, I am persuaded, no less useful even for the proof of the theorems themselves; for certain things first became clear to me by a mechanical method, although they had to be demonstrated by geometry afterwards because their investigation by the said method did not furnish an actual demonstration."
Translation by T.L. Heath

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What does this tell us?

- Can certainly say that Archimedes anticipated many ideas of *integral calculus* in this work
- But he was also *careful* and realized that he didn't have a complete justification for the idea of "balancing" an area with a collection of line segments
- So he also worked out the first "method of exhaustion" proof to supply an argument that would satisfy the mathematicians of his day

Recognized in antiquity as a master

"In weightiness of matter and elegance of style, no classical mathematics treatise surpasses the works of Archimedes. This was recognized in antiquity; thus Plutarch says of Archimedes' works:
'It is not possible to find in all geometry more difficult and intricate questions, or more simple and lucid explanations. Some ascribe this to his genius; while others think that incredible effort and toil produced these, to all appearances, easy and unlaboured results.' "

A. Aaboe, *Episodes from the early history of mathematics*

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- Also extremely poignant to think what might have been if others had been better prepared to follow his lead at the time!