The Eclectic Content and Sources of Book IV of Clavius' *Geometria Practica*

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Overview

- I've always been interested in the history of mathematics (in addition to my nominal specialty in algebraic geometry/computational methods/coding theory, etc.)
- To pursue this historical work, I've been studying Latin and Greek with our excellent (and very welcoming) Classics department
- The subject for today relates to a Latin-to-English translation project I am currently pursuing – working with the *Geometria Practica* of Christopher Clavius, S.J. (1538 - 1612, CE)

Christophorus Clavius, S.J. 1538 - 1612

- *Euclidis Elementorum, Libri XV* (1574)
- *Epitome artithmeticae practicae* (1583)
- *Geometria practica* (1604)
- *Algebra* (1608)
- Commentary on the *Sphere* of Sacrobosco (astronomy), other textbooks
- Chief mathematician on Gregorian calendar reform

Context of Clavius' work

- A long tradition of books with Islamic and Greek roots, starting in the late Medieval period (e.g. Leonardo Pisano – "Fibonacci", 1170 - 1240 CE *De Practica Geometriae*)
- "Orthogonal to" the Platonic orientation of Euclid in some ways:
  - Discussed how to make and use physical instruments – astrolabe, quadrants, proportional compasses, etc.
  - Intended to be *practically useful* for land surveying, architecture, engineering, military science, etc.
  - Used *numerical measures of lengths and angles*; mensuration formulas; numerical examples in addition to, or instead of, Euclidean proofs

Book 4 of Clavius' *Geometria Practica*

- Overall topic – *measuring plane areas*
- Rectangles, triangles, nonrectangular quadrilaterals, general polygons, then
- Regular polygons, circles, and finally
- Sectors and segments of circles, and,
- (as an "appendix"), ellipses and segments of parabolas (after Archimedes, from *Conoids and Spheroids*, *Quadrature of the Parabola*)

Some results are only quoted

- (p. 202) "It is pleasing to conclude this fourth book with two theorems most acutely discovered and proved by Archimedes the Syracusan."
- The first (p. 202): If *ABCD* is an ellipse with minor diameter *AC* and major diameter *BD*, then let *HI* be the mean proportional between *AC* and *BD*. The area of the ellipse is equal to the area of the circle with diameter *HI*.
- The second (p. 203): If *ABC* is a segment of a parabola with vertex *B*, and base *AC*, then the ratio of the area of the segment to the area of the triangle is $\frac{4}{3}$ ("sesquitertia")
- But Clavius doesn't give the proofs (in effect, they are much more advanced than most of the topics he includes)

Others are proved in detail

- Earlier in Book 4, Clavius gives a reworked version of essentially all of what survives of Archimedes' *Measurement of the Circle*
- "It will not be a digression, therefore, if I include this truly most acute and precise book on the measurement of the circle, partly because it is very brief ... , partly so that the student, in order to understand something so useful and so widely disseminated by other masters, should be encouraged to go to Archimedes himself, and finally mostly because the writings of Archimedes, as a result of their brevity, are somewhat more obscure, and we hope to be able to bring some light to them. Nor do we doubt that these things will be welcome and enjoyable for a studious reader."

A first result

Clavius and Archimedes begin by showing this result:

Theorem (*Measurement of the Circle, Prop. 1)*

The area of any circle is equal to the [area of] a right triangle, one of whose sides about the right angle is equal to the semidiameter (i.e. radius) of the circle, while the other is equal to the circumference of the same circle.

The proof is given in full by the classical "method of exhaustion" in the form perfected by Archimedes.

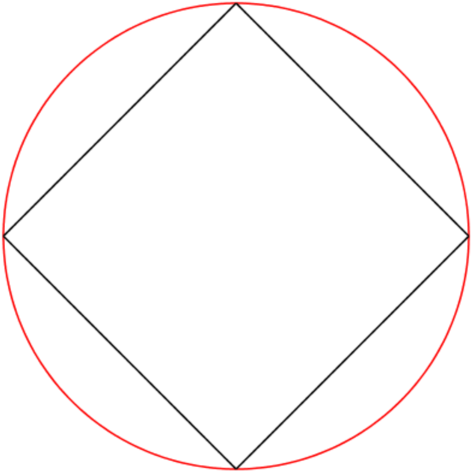
Background – The "Archimedean Principle"

- Clavius (and less explicitly, Archimedes) make use of the following. I'll use modern notation for simplicity:
- Let $\varepsilon > 0$ be arbitrary. *If* $A_1 > A_2 > A_3 > \cdots$ *is a sequence of areas, in which* $A_{n+1} < \frac{1}{2} A_n$ *for all* n*, then for some* n_0*,* $A_{n_0} < \varepsilon$*.*
- An easy consequence of the standard Archimedean principle and the unboundedness of the sequence 2^n.

An Archimedean proof “by exhaustion”

- First suppose the circle is larger than the stated triangle by a certain positive magnitude (i.e. area – Clavius calls this $z > 0$)
- A sequence of non-overlapping areas is removed from the interior of the circle.
- Specifically, the inscribed square is removed at the first step, then four triangles on the sides of the square inscribed in the circle, so a regular octagon has been removed, then eight triangles on the sides of the octagon, so the total removed figure is a regular 16-gon, etc.

What this looks like



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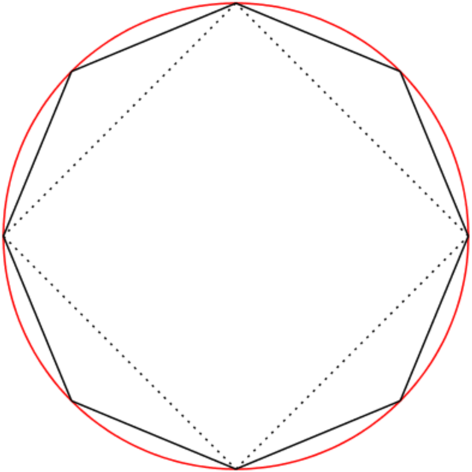

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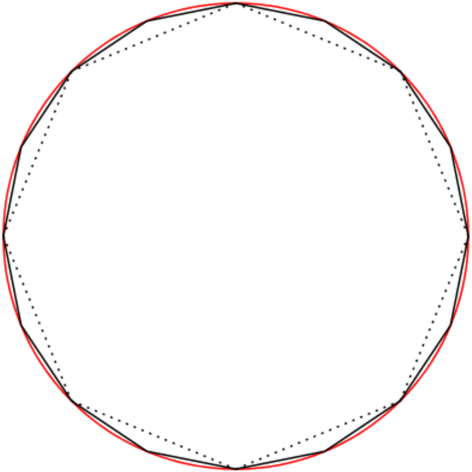


Exhaustion proof, cont.

- The "*Archimedean Principle*" implies that eventually the remaining area will be less than z, since at each stage, more than half of the area left at the previous stage is removed (a nice trigonometric exercise!)
- But this leads to a contradiction for n_0 sufficiently large: The area of the circle is supposed to equal $(\text{area of triangle}) + z$, but the area of the circle also equals:

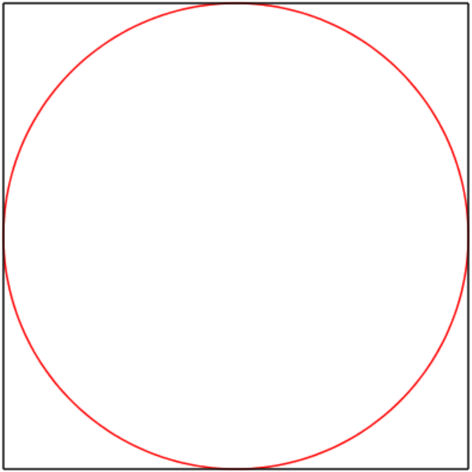
$$(\text{area of polygon}) + (\text{remaining area}) < (\text{area of triangle}) + z$$

  (since the perimeter of the polygon is smaller than the circumference of the circle and the apothem is less than the radius)
- Hence the area of the circle cannot equal the area of the triangle $+z$.

Exhaustion proof, cont.

- Now suppose that the circle is less than the stated triangle by a certain magnitude (again denoted z)
- Now Archimedes starts from the circumscribed square about the circle (area definitely greater than the triangle since the perimeter of the square is greater than the circumference of the circle, and the apothem is the same as the radius).
- Now begin removing areas from the square (first the circle), then triangles with base tangent to the circle. The remaining regions now are collections of "mixed triangles" with one side an arc of the circle.

What this looks like



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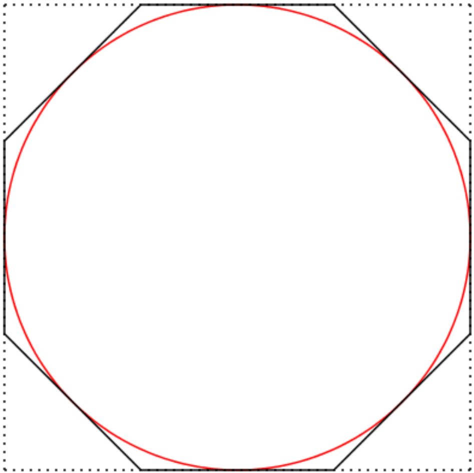

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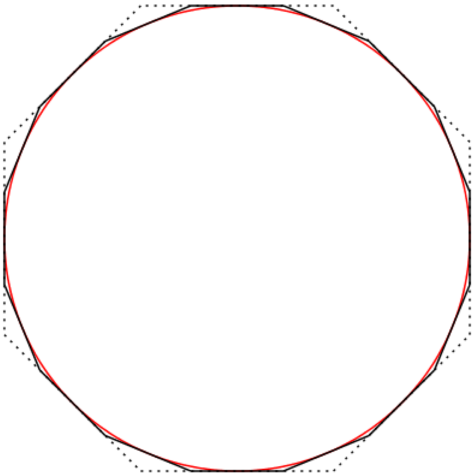

Exhaustion proof, concluded

- Again, more than half the remaining area is removed at each step, so eventually the remaining area becomes less than z.
- This also leads to a contradiction, since both

$$(\text{area of polygon}) > (\text{area of triangle})$$

and

$$(\text{area of polygon}) < (\text{area of circle}) + z = (\text{area of triangle})$$

- Hence since the triangle is neither greater nor less than the circle, it can only *equal the circle*. QED

An amusing sidelight

In a "scholium" to this proposition, Clavius takes the French Protestant humanist Joseph Justus Scaliger (1540-1609) to task. Scaliger was an eminent classicist and literary scholar who also fancied himself a brilliant mathematician. His mathematical *magnum opus* was grandly titled *Cyclometrica*, and published in 1594. (The 1st edition is lavishly printed with statements of theorems in both Latin and ancient Greek!) Scaliger claimed this proof by Archimedes was incorrect (what he said was essentially, amazingly enough, knowing $a > b$ and $a < b$ are both false does not show $a = b$!) He went on to "prove" that a circle has $\frac{6}{5}$ the area of the inscribed regular hexagon, i.e. $\pi = \frac{9}{5}\sqrt{3} \doteq 3.11769$. This "implied" the circle was squared with straightedge and compass!

Scaliger and Clavius

Scaliger had a number of other "run-ins" with Jesuits on various subjects; Clavius' satisfaction at being able to point out his gross errors is palpable in this passage:
"And I am astonished, Mathematicus, that you want to deny that some quantity is equal to another when it is neither greater nor less. For if it is not equal [to the other], then it will be unequal [to the other], therefore either greater or less, against that hypothesis. Or don't you see that not only Archimedes, but also Euclid used this way of arguing most frequently in Book XII?"
In public Scaliger said something gracious(?) to the effect that he would rather be criticized by Clavius than praised by other men, but he also privately lampooned Clavius for his appetite for food and drink ("c'est un gros ventre d'Aleman").

Continuing with Archimedes *Measurement of the Circle*

- The remainder of Archimedes' work and Clavius' treatment of it is devoted to a painstaking numerical computation of approximations to π, the ratio between the circumference and the diameter of a circle.
- By considering repeatedly bisecting angles in circum- and inscribed squares until he reaches regular 96-gons, Clavius, following Archimedes gives these estimates: $\frac{223}{71} = 3\frac{10}{71} < \pi < \frac{22}{7} = 3\frac{1}{7}$.
- This sort of numerical calculation involved was revolutionary in Greek geometry. Its usefulness was recognized immediately and this work of Archimedes was justly famous.

More accurate approximations

Somewhat later, on page 198, Clavius gives the following, quite amazing figures for a circle:

Diameter: 100000000000000000000.

Circumference, overestimate: 314159265358979323847.

Circumference, underestimate: 314159265358979323846.

π to 20 decimal places in 1604(!)

- These approximate values were computed by the Dutch calculator Ludolph van Ceulen (1540 - 1610) (who also had run-ins with Scaliger), and Clavius gives him credit, mentioning also his own younger colleague, Christopher Grienberger, S.J. (1561-1638).
- " ... while the difference between the true area and the computed area may be very small for small circles [using the Archimedean ratios], yet in larger circles it may not be negligible. If one should desire more accurate areas of circles, one may take the closer approximations [above]. "

Something of a surprise

- Clavius *begins* his discussion of areas of triangles by discussing the following “rule:”
- *Add all the sides together in one sum; each of the sides are subtracted from half of this sum, so that three differences between the sides and [the semiperimeter] are obtained; finally, these three differences and the semiperimeter are multiplied together. The square root of the number produced will be the area of the triangle which is sought.*
- Today known as *Heron's formula*: If a, b, c are the sides of a triangle and $s = \frac{a+b+c}{2}$, then the area is

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

- In addition to the “rule” and numerical examples, Clavius does give a complete, Euclidean-style proof.

A perplexing fact

- Clavius doesn't attribute this to anyone (unlike many other things such as the treatment of Archimedes described above).
- Heron, sometimes written as Hero, is Heron of Alexandria (ca 10 - 70 CE(?)), a fascinating figure in Greek engineering, science, and mathematics.
- He, even more so than Archimedes, was a sort of "anti-Euclidean" figure interested in mechanical inventions and applications of geometry.
- The formula appears in Heron's *Metrica*, but this work was considered lost *until 1896*, when R. Schöne found a manuscript copy in Istanbul.

Tracking down Clavius' source

- Clavius' proof itself is equivalent mathematically to the proof in Heron's *Metrica*, but there are differences (mostly filling in steps and giving direct arguments rather than quoting results about the inscribed circle of the triangle and cyclic quadrilaterals).
- The actual text is quite a bit closer (i.e. extremely close; virtually word for word) to the proof of the same result in Leonardo of Pisa's (i.e. Fibonacci's) *De Practica Geometriae* (!)

Moving back in time

- M. Clagett (in *Archimedes in the Middle Ages*, v I, p. 224) says that Fibonacci's source was the Latin "Verba filiorum Moysi filii ... " or "Liber trium fratrum de geometria" by Gerard of Cremona (1114 - 1187 CE)
- From Arabic book on mensuration by the three "Banu Musa" brothers in 9th century Baghdad (sons of Musa ibn Shakir) – also includes Archimedes' results from *Measurement of the Circle!*
- The Banu Musa were translators at the “House of Wisdom” in Baghdad engaged in translating Greek mathematical works into Arabic (and other scientific pursuits!)
- Gerard's version was the first proof of Heron's formula in Latin

Questions!

- Clagett says that the proof of Heron's formula in Fibonacci is essentially copied directly from the translation of the book by the Banu Musa.
- (I haven't been able to verify this directly yet, but I'm willing to believe him!)
- The Banu Musa probably knew Heron's *Metrica*.
- A later Islamic mathematician (al-Biruni) thought the result ultimately came from Archimedes and this conjecture has sometimes been treated as a fact (for instance by Heath in his *History of Greek Mathematics*). No extant work of Archimedes does this, though.

More questions!

- Interestingly enough, there are some features of the proof that are perplexingly foreign to standard Greek geometry (like Euclid, or Apollonius, Archimedes?)
- Especially: multiplying four lengths together and interpreting that as a geometric magnitude (the square of an area).
- Some of Clavius' argument, in fact, also tries to deal with this by saying that the proof is working with numbers representing the lengths, rather than with the lengths themselves, etc. (but that's a later point of view!)

Conclusions

- So the jury is still out, so to speak, about how Archimedes or some other pre-Heron Greek might have thought about the proof, although there have been a number of attempts at reconstructions.
- My guess: Clavius either had Fibonacci or Gerard of Cremona's book at hand and followed that.
- But why didn't he attribute this to either of them??

References

[1] Barnabas Hughes, ed. *Fibonacci's De Practica Geometrie*, Springer, NY, 2008.

[2] Christophorus Clavius, S.J., *Geometria practica* (1606 edition), online at `http://www.e-rara.ch`

[3] Marshall Clagett, *Archimedes in the Middle Ages*, vol I. University of Wisconsin Press, Madison, 1964.