Mathematics and Music

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A Greek perspective

- Archytas (a Pythagorean, contemporary of Plato): "*Mathematicians seem to me to have arrived at correct conclusions, and it is not therefore surprising that they have a true conception of the nature of each individual thing ... Thus they have handed down to us clear knowledge about the speed of the stars, their risings and settings, and about geometry, arithmetic, ... and last, not least, about music, for these seem to be sister [sciences]*."

What is it about math and music?

- Mathematics and music have been associated throughout history.
- Mathematicians and others have felt the affinities, even if they have not been able to pin down exactly what it is about the subjects that ties them together.

“Obvious” and “Non-obvious” connections

- Physical vibrations and the sound waves they produce – tied to solutions of wave equations ($u_{tt} = c^2 u_{xx}$ for vibrating strings, organ pipes, etc.)
- *Number symbolism* in music.
- Mathematical descriptions of proportion and form (e.g. golden ratio).
- Affinities between the ways mathematicians think and the ways musicians think about the “materials” of music.

Plan for this Talk

- Organization of music
- The time and pitch dimensions
- The language of symmetry
- The topology/geometry of “musical space”
- Two extended examples
  - J. S. Bach's *The Musical Offering*
  - Nzakara harp music

Organization of music

- Pieces of music are composed and experienced as time-sequences of “events” or “gestures”
  - melodies or motifs
  - harmonic progressions
  - rhythmic patterns, etc.
- organized in various ways, using
  - repetition
  - grouping
  - variation

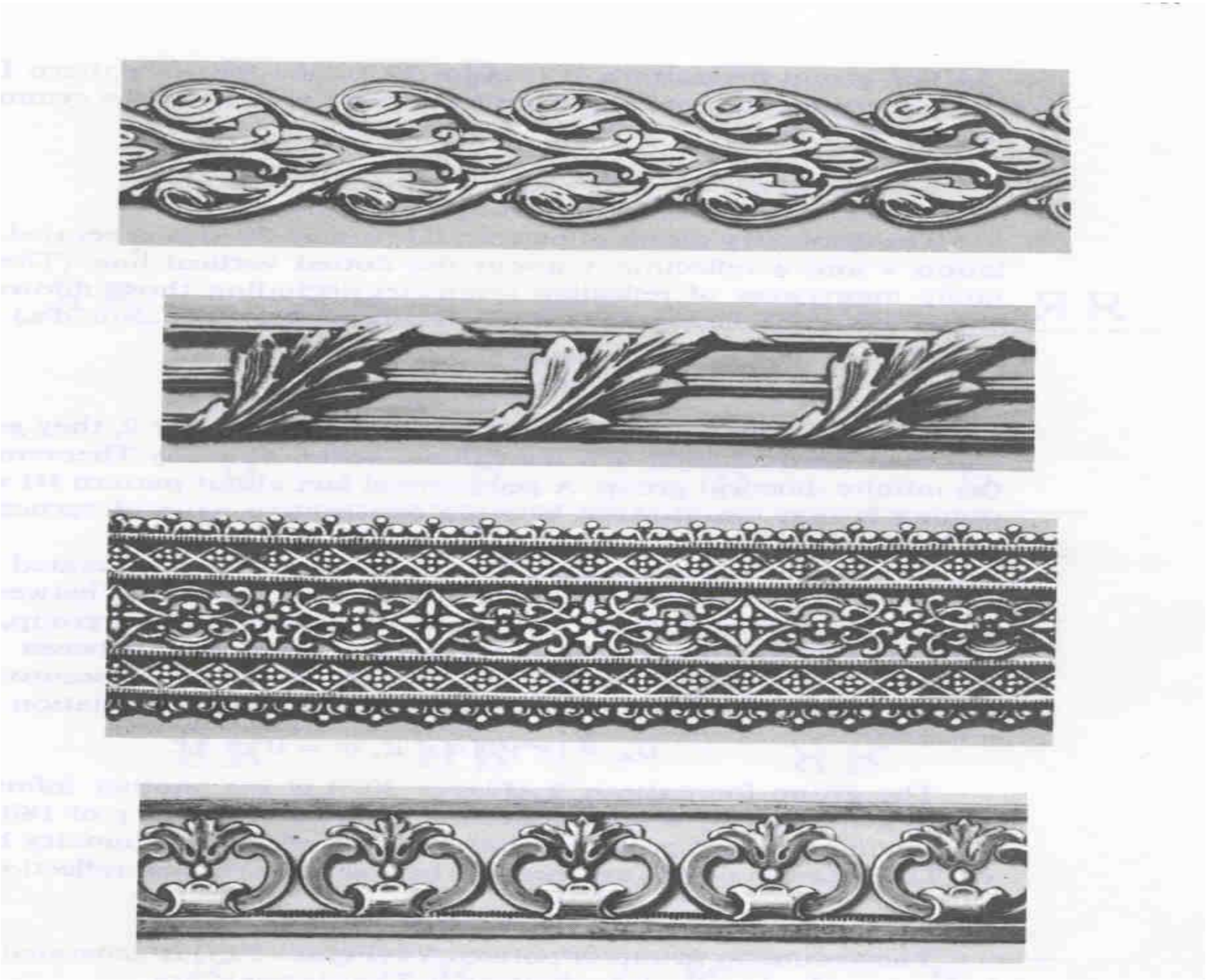
Describing musical forms

- Ternary song form: ABA
- Verse, refrain song form: VRVRVR ... (any number of verses)
- Sonata allegro form: Exposition, Development, Recapitulation
- Rondo form: ABACABA (one type, even more episodes sometimes occur)
- Variation forms (repetitions of a basic pattern, but altered on each repetition)

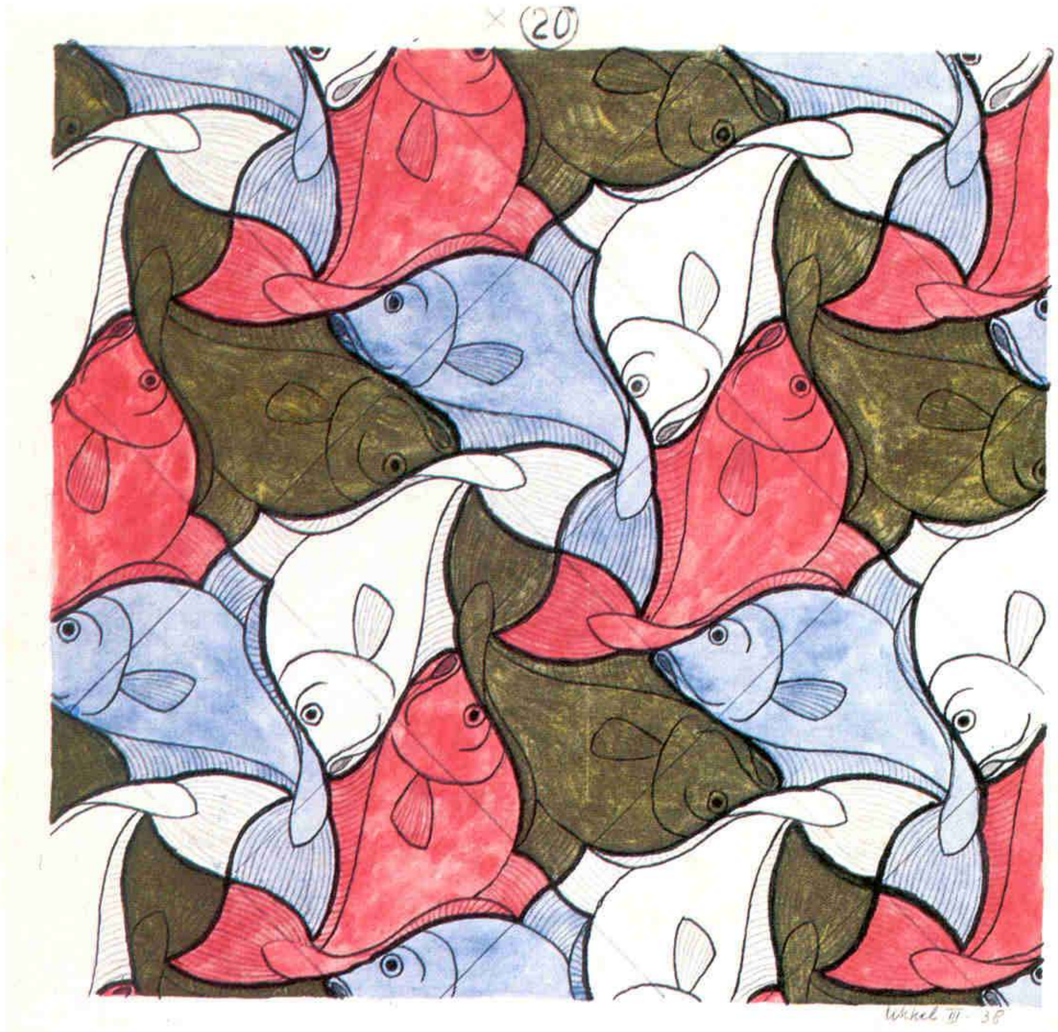
Balance and symmetry

- Note that some of these are very "symmetrical" (ABA, ABABCBA)
- Using ABA form gives a feeling of balance or closure at end -- we return to the original music after a varied middle section.
- Forms have close analogs in visual arts too (e.g. frieze patterns, tesselations)

Visual analogies



Another visual analogy



Mathematical symmetry

- The precise mathematical understanding of symmetry: *invariance under a transformation.*
- For instance, bilateral symmetry is invariance under reflection across a plane in three dimensional space (or across a line in the plane).
- Other types of transformations can be considered and other symmetries studied.
- Geometrical examples: translations, rotations.
- Each gives a corresponding form of symmetry.

How does this apply to music?

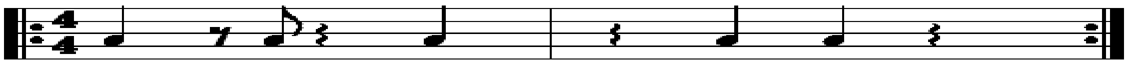
- The idea is that various “dimensions” of a musical composition
  - the points in time when different events occur
  - duration and spacing of events
  - the pitches that sound
  - others, too(!)
- can also be subjected to various types of transformations.

Symmetry in Music

- These transformations can be used by composers to build up pieces locally and even globally.
- A musical composition can exhibit the kind of invariance (or “idealized” invariance) that we see in frieze patterns or tesselations such as the Escher drawing from before.

The time dimension and rhythm

- Western music is represented by a notational system that has some common features with mathematical *graphs*.
- Reading notation left to right gives the time-sequencing of events. For instance, here is a representation of the 3-2 *son clave* rhythmic pattern that forms the underpinning for salsa dance music*:*



One form of symmetry in time

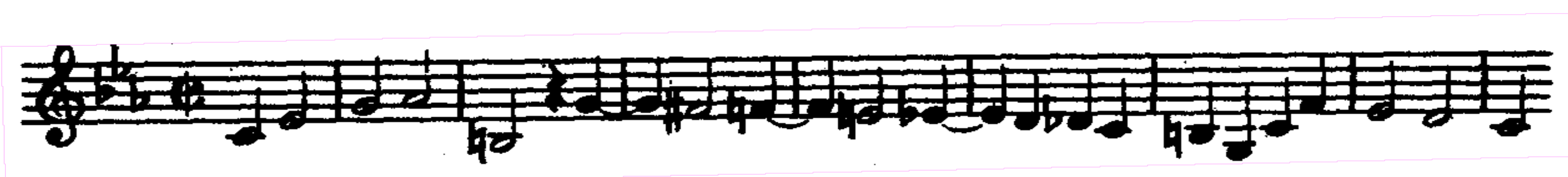
- For long stretches of time in salsa music, this pattern repeats, unchanged (underneath rest of the music)
- So, if we translate forward or backward in time by the length of time needed to perform the basic pattern (hence any integer multiple of them), then that rhythmic structure would be unchanged – it is invariant (periodic) under time translations.
- Of course a real piece of music has a beginning and an end in time, so this is a mathematical idealization of the actual properties of the music.

Another form of time symmetry

- Another type of transformation in time is analogous to *reflection.* Namely, we could imagine reflecting a piece of music about its midpoint in time. This would have the effect "running time in reverse" and transforming the sequence of events in a piece of music accordingly.
- Symmetry under this sort of transformation would be similar to the property of *palindromes* in language: "able was I ere I saw Elba".

The pitch dimension

- Here is the notation for a theme from J. S. Bach's *A Musical Offering* that we will hear in a bit.
- Symbols at left are the clef and key signature, which set up the correspondence between lines and spaces of the staff and pitches.
- Durations of notes indicated as before.



Describing pitch mathematically

- The way we perceive pitch is periodic in a *logarithmic* sense – *doubling* the frequency of sound waves yields notes that differ by the interval called the *octave*.
- The *many* ways different musical traditions and tuning systems subdivide the octave into intervals is a subject for another talk(!)

Equal Temperament

- Subdivide into 12 “equal” subintervals.
- Traditional names, starting from C, are: C,C#=D-flat,D,D#=E-flat,E,F,F#=G-flat,G,G#=A-flat,A,A#=B-flat,B,C.
- Can associate these with integers *mod 12* 0,1,2,3,4,5,6,7,8,9,10=t,11=e.
- Study geometry of “pitch space” (see D. Tymoczko, ``Geometry of Musical Chords'' *Science* **313** (2006), 72-74).

"*Geometry of Musical Chords*"

- All pitch classes (including microtones) = elements of $\mathbf{R}/12\mathbf{Z}$ (a circle)
- Chords can be represented by either ordered or unordered *n*-tuples of pitch classes.
- Examples: C major chord {C,E,G} = {0,4,7}. Dominant 7th chord in D major {D,F#,A,C} = {2,6,9,0}, etc.

Transposition

- *Translating* up or down in pitch is a very common operation known as *transposition.* (Often done to facilitate performance in a different pitch range than originally intended.)
- Example: F major Chord {F,A,C} = {5,9,0} comes from the C major chord by adding 5 mod 12.
- All major chords are transpositions of each other.

Inversion

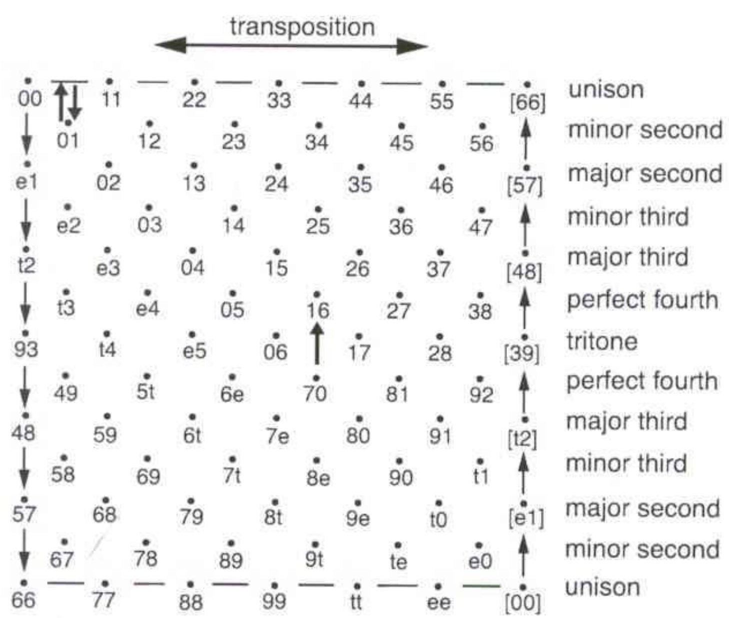
- Another common musical transformation is another type of reflection – reflection across a fixed pitch level.
- Example: The C minor chord {C,E-flat,G} = {0,3,7} is obtained from the C major chord {C,E,G} = {0,4,7} by the following reflection: $x \to 7 - x \bmod 12$ (fixed point is a quarter tone halfway between E and E-flat.)

Chord Space

- Distinguishable voices -- *n*-note chords correspond to elements of the torus $\mathbf{T}^n = (\mathbf{R}/12\mathbf{Z})^n$
- Indistinguishable voices -- *unordered n-tuples,* elements of the quotient space $\mathbf{T}^n/S_n$ where S_n is the symmetric group on *n* letters.
- This is an "orbifold" (quotient of a manifold under a finite group action).

The space of 2-note chords

- topologically: a Möbius strip with singular edge (along fixed points – repeated notes)



Voice leadings, harmony etc.

- The harmonic progression of a piece of music can be viewed as a path in the corresponding chord space.
- Tymoczko has some fascinating movies of familiar pieces showing how composers have used the geometry (intuitively) to find modulations between different keys in tonal music, etc.

Two extended examples

- We have been discussing possible transformations of music and symmetry properties from a more or less “theoretical” point of view so far.
- The next goal is to explore two extended examples to see some of these ideas “in action”.
- The first piece we will look at is called *A Musical Offering,* by Johann Sebastian Bach (1685-1750). There's an interesting story attached to the creation of this piece.

The *Musical Offering* canons

- We'll concentrate on several of the canons that make especially strong use of musical transformations and symmetry. Here is the notation of one:

Structure of this canon

- The piece is 18 measures long (played twice on recording).
- In measures 1 – 9, the top voice plays the Royal theme, while the bottom voice plays an accompanying figure, with faster motion.
- In measures 10 – 18, the top voice switches to the accompanying figure, while the lower voice plays the royal theme, *but both are moving backwards in time.*
- *Also called a canon cancrizans = "crab canon"*

Structure, continued

- In other words, this canon is a musical "palindrome" -- as a whole piece of music, it is symmetric under reflection in time around the barline between measures 9 and 10. It would sound the same played forwards or backward in time(!)
- Another point: Who's Kirnberger? We didn't mention this before, but one curious feature of Bach's *Musical Offering* was that the canon sections were actually left as puzzles for other musicians to decipher. Kirnberger was the solver!

The canon at the unison

- What symmetry is involved here?

The modulating canon

- "As the notes rise, so may the glory of the King" -- was Bach being ironic?

The canon by contrary motion

- What symmetry transformation is involved here?

Some food for thought

- The earliest known examples of canons in medieval music definitely have an element of religious symbolism -- the first voice sets out The Law in musical terms and the other voices follow obediently.
- What would you expect music written to flatter a royal patron and extol his "greatness" to sound like? Do these pieces sound like that?

Symmetry in Nzakara music

- The Nzakara are an ethnic group living in a region split between Central African Republic, Sudan, and Zaire. The have a long musical tradition (which is now sadly on the verge of dying out).
- Work of Marc Chemillier at IRCAM in Paris has brought to light some of the remarkable structures underlying the harp and xylophone music that was performed to accompany poetry at the courts of their kings before the arrival of Europeans.

The Nzakara Harp

- The Nzakara music we will look at was performed on a five-stringed harp (notes sounding roughly C, D, E, G, B-flat).
- Here's a picture of one of the last performers with his instrument:

A completely oral tradition

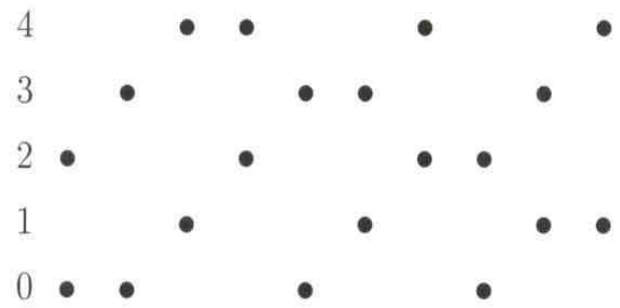
- Interestingly enough, the Nzakara transmitted their musical compositions and performance practice orally – they had no system of musical notation like the one we have seen in the examples we have studied so far.
- So we will use a method for describing their music developed by ethnomusicologists. Much more information on this can be found in the web site and publications of Marc Chemillier.
- Also see the upcoming book *Music, A Mathematical Offering* by Dave Benson for more details.

Nzakara music

- The Nzakara harpists performed accompaniments to sung poetry consisting of patterns of notes repeated in the same form many times (basic translation symmetry in time). But there was a great deal of additional structure:
  - the five pitches of the harp were viewed in cyclic arrangement (call these 0,1,2,3,4, so 0 follows 4).
  - only 5 different combinations of two pitches were ever plucked simultaneously: (1 and 4, 0 and 2, 1 and 3, 2 and 4, or 0 and 3)

Nzakara music, continued

- Depending on the poem, one of a catalog of basic different patterns was played.
- Here is a representation of one of them, called an *ngbakia:*



Representing the pattern

- Let's introduce the following labels for the combinations of two strings sounding simultaneously: 1 and 4 – label 0, 0 and 2 – label 1, 3 and 1 – label 2, 4 and 2 – label 3, and 3 and 0 – label 4. (These are cyclically ordered in the same way the pitches of the harp strings are ordered.)
- Then the repeating *ngbakia* pattern corresponds to the sequence of labels*: 1403423120*

Surprising structure

- There's a rather amazing amount of structure in this and in all of the Nzakara patterns that have been studied.
- Divide the label string into two halves:

  14034 23120

- Do you notice something about the first and second halves? (Can you transform one into the other some way?)
- What does this say about the *ngbakia?*

Questions for further thought

- Not all musicians use these ideas, and even those who do don't *always* use them. What might be a reason for incorporating these ideas in a piece?
- When they do, are they "doing mathematics"?
- Is this *limited* to mathematics and music? Are we perhaps saying something about human intelligence in general?
- Knowing what you do now, let's return to the Haydn Minuet.

The Haydn Minuet